Mathematics for Economics and Business

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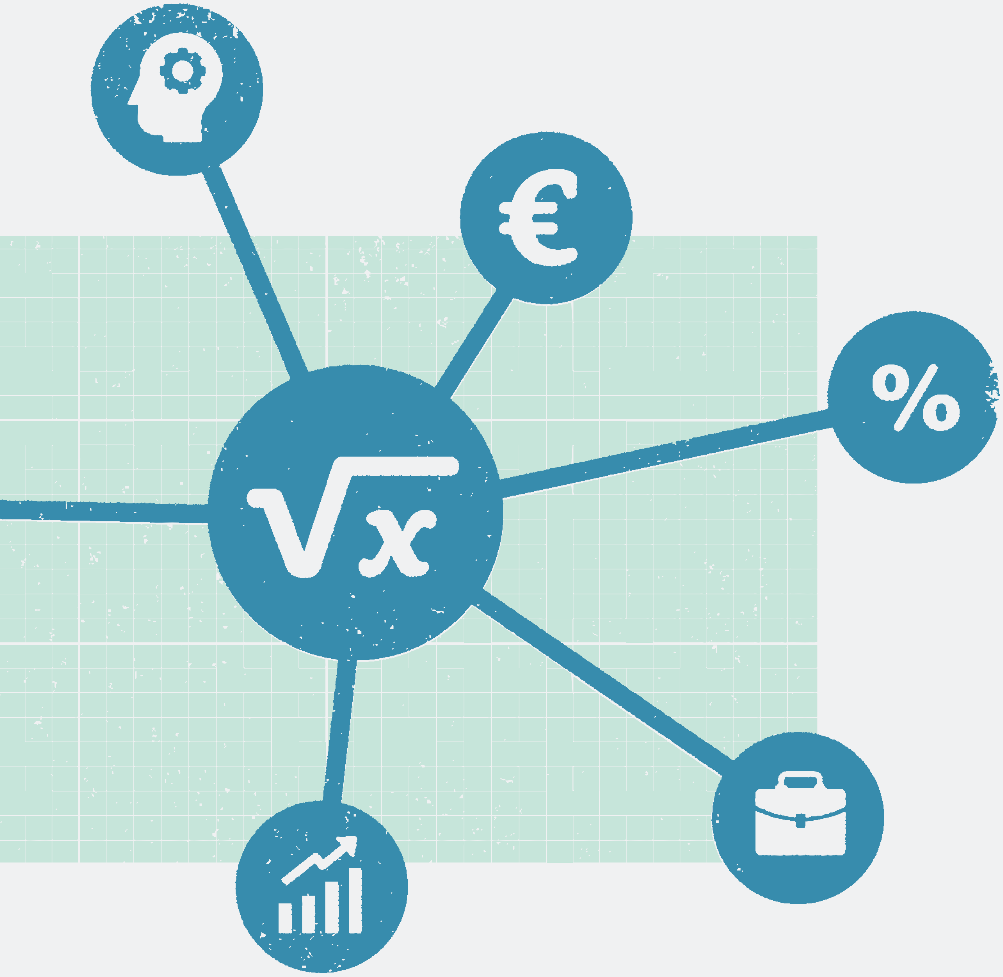

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1 Matrices, Determinant and Rank

The most simple use of a matrix is to organise data. From this point of view a matrix is nothing but a cabinet with drawers where we store information. However, in mathematics matrices are useful for many other reasons.

Even though in this chapter we will only study what a matrix is and the main concepts related to it, the student will discover multiple applications of matrices during the course of study. The most well known application is to solve systems of linear equations. Other examples include linear programming, solving of differential equations and the optimisation of functions of several variables.

1.1. Matrix and Dimensions of a Matrix

Definition 1 ***(Matrix and dimensions).*** A **matrix** is a set of numbers arranged in rows and columns. The number of rows (m) and the number of columns (n) constitute the **dimensions** of a matrix, which will be denoted by $m \times n$.

Usually an $m \times n$ matrix is represented as follows:

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

where a_{ij} stands for the element located at row "i" and column "j" of the matrix A.

Example 1. The matrix $A = \begin{pmatrix} 6 & 3 & 4 \\ 8 & 2 & 5 \end{pmatrix}$, that has two rows and three columns, is said to be a 2×3 matrix.

Example 2. A company has three production plants, p_1, p_2 and p_3, and uses raw materials from 10 different suppliers $z_1, z_2, \ldots, z_{10}$. Using a 3×10 matrix we can represent the quantities (described in thousands of kg, for instance) that each production plant uses from every supplier. The result could be

$$A = \begin{array}{cc} \begin{matrix} z_1 & z_2 & z_3 & z_4 & z_5 & z_6 & z_7 & z_8 & z_9 & z_{10} \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \end{matrix} & \\ \begin{pmatrix} 3 & 5 & 4 & 0 & 0 & 0 & 1 & 2 & 2 & 9 \\ 2 & 3 & 6 & 2 & 2 & 0 & 2 & 7 & 8 & 9 \\ 1 & 1 & 3 & 1 & 2 & 10 & 10 & 10 & 17 & 1 \end{pmatrix} & \begin{matrix} \leftarrow p_1 \\ \leftarrow p_2 \\ \leftarrow p_3 \end{matrix} \end{array}$$

As an example, $a_{23} = 6$, indicates that the production plant p_2 has used 6000 kg from supplier z_3.

Types of Matrices

By using the concept of dimension of a matrix, we can give a first classification of matrices:

- A **square matrix** is a matrix with the same number of rows and columns, that is, an $n \times m$ matrix where $m = n$. In a square matrix, we call the **main diagonal** the set of elements $a_{11}, a_{22}, \ldots, a_{nn}$.
- A **diagonal matrix** is a square matrix in which all the elements outside the main diagonal are zero.
- A **rectangular matrix** is a matrix in which the number of rows is different from the number of columns, that is, $m \neq n$.
- A **row vector** is a $1 \times n$ matrix, that is, a matrix with a single row.
- A **column vector** is an $m \times 1$ matrix, that is, a matrix with a single column.

Example 3:

- The matrix

$$A = \begin{pmatrix} 4 & 3 \\ 1 & 5 \end{pmatrix}$$

is a 2×2 square matrix. In general, an $n \times n$ square matrix is said to be a matrix of dimension n.

- Given the matrix

$$A = \begin{pmatrix} 3 & 3 & 5 \\ 1 & 0 & 2 \\ 3 & 1 & 8 \end{pmatrix}$$

its main diagonal consists of the elements a_{11}, a_{22} and a_{33}. Hence, the main diagonal is $\{3, 0, 8\}$.

- The matrix $A = \begin{pmatrix} 6 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 8 \end{pmatrix}$ is a diagonal matrix.

- The matrix $A = \begin{pmatrix} 2 & 3 & 4 \\ 5 & 2 & 1 \end{pmatrix}$ is a rectangular matrix whose dimensions are 2×3.

Exercises

1. Find the dimensions of the following matrices:

$$A = \begin{pmatrix} 1 & 1 & 2 & 2 \\ -2 & 1 & 3 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 8 & 8 \\ 7 & 0 \end{pmatrix} \quad C = \begin{pmatrix} 2 & 2 & 1 \end{pmatrix}$$

Taking into account the dimensions, identify the type of each matrix.

2. Compute the value of x for which the matrix $A = \begin{pmatrix} x & x-1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix}$ is diagonal.

1.2. Operations with Matrices

1.2.1. Addition of Matrices

Definition 2 ***(Addition of matrices).*** Given two matrices of the same dimensions, A and B, their **sum** is a new matrix of the same dimensions that is obtained by adding the elements located at the same positions in the two matrices.

Properties of the addition of matrices

The addition of matrices of the same dimensions satisfies the following properties:

- Associativity: $(A + B) + C = A + (B + C)$
- Commutativity: $A + B = B + A$
- Existence of the identity element: $A + (0) = A$
  The zero matrix (0) is the matrix of the same dimensions as A, in which all the elements are 0.
- Every matrix A has a symmetric matrix: $A + (-A) = (0)$
  The symmetric matrix, $(-A)$, is the matrix obtained by changing the sign of every element in the matrix A.

Example 4. A paper factory manufactures two different types of paper, p_1 and p_2, that are sold to three different communities, c_1, c_2, and c_3. The quantities of product that are sold weekly, in thousands of kg, can be represented by using a 2×3 matrix. Suppose that during the last two months the number of items sold have been

$$\begin{array}{cc} & \begin{array}{ccc} c_1 & c_2 & c_3 \\ \downarrow & \downarrow & \downarrow \end{array} \\ \begin{array}{c} p_1 \rightarrow \\ p_2 \rightarrow \end{array} & \begin{pmatrix} 2 & 5 & 0 \\ 3 & 1 & 2 \end{pmatrix} \end{array} \quad \text{and} \quad \begin{array}{cc} & \begin{array}{ccc} c_1 & c_2 & c_3 \\ \downarrow & \downarrow & \downarrow \end{array} \\ \begin{array}{c} p_1 \rightarrow \\ p_2 \rightarrow \end{array} & \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 2 \end{pmatrix} \end{array}$$

In order to calculate the total sales in the last two months, we have to add the matrices that indicate the monthly quantities. In this way, we obtain

$$\begin{pmatrix} 2 & 5 & 0 \\ 3 & 1 & 2 \end{pmatrix} + \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 2+1 & 5+2 & 0+3 \\ 3+2 & 1+1 & 2+2 \end{pmatrix}$$

That is, the number of items of each type that were sold in the last two months are described using matrix notation by

$$\begin{array}{cc} & \begin{array}{ccc} c_1 & c_2 & c_3 \\ \downarrow & \downarrow & \downarrow \end{array} \\ \begin{array}{c} p_1 \rightarrow \\ p_2 \rightarrow \end{array} & \begin{pmatrix} 3 & 7 & 3 \\ 5 & 2 & 4 \end{pmatrix} \end{array}$$

1.2.2. Subtraction of Matrices

Definition 3 ***(Subtraction of matrices).*** Given two matrices of the same dimensions, A and B, the **subtraction** $A - B$ is defined based on the sum and the concept of symmetric matrix introduced above. The subtraction $A - B$ is defined by

$$A - B = A + (-B)$$

1.2.3. Scalar Multiplication of a Matrix

Definition 4 ***(Scalar multiplication of a matrix).*** Given a real number λ and a matrix A, the **multiplication** $\lambda \cdot A$ is a matrix of the same dimensions as A that is obtained by multiplying each element of the matrix by the real number λ.[1]

Properties of the Scalar Multiplication of a Matrix

- Associativity: $(x \cdot y) \cdot A = x \cdot (y \cdot A)$
- Distributivity 1: $(x + y) \cdot A = x \cdot A + y \cdot A$
- Distributivity 2: $x \cdot (A + B) = x \cdot A + x \cdot B$
- Existence of the identity element: $1 \cdot A = A$

Example 5. A given distributor has demanded the prices for two different models, m_1 and m_2, of high definition screens from a producer in the USA. The price of each model depends on the quality of the finished goods and there are three different qualities, q_1, q_2 and q_3, for each of the two models. The dollar prices have been presented in the following matrix:

$$\begin{array}{cc} & \begin{array}{ccc} q_1 & q_2 & q_3 \\ \downarrow & \downarrow & \downarrow \end{array} \\ \begin{array}{c} m_1 \rightarrow \\ m_2 \rightarrow \end{array} & \begin{pmatrix} 1300 & 1200 & 1000 \\ 2500 & 2300 & 2200 \end{pmatrix} \end{array}$$

1 The term *scalar* denotes a real number.

In order to transform the prices to euros, we have to take into account the exchange rate. Suppose for instance that €1 = \$1.6, then the prices matrix in the European currency would be

$$\frac{1}{1.6} \cdot \begin{pmatrix} 1300 & 1200 & 1000 \\ 2500 & 2300 & 2200 \end{pmatrix} = \begin{pmatrix} \frac{1300}{1.6} & \frac{1200}{1.6} & \frac{1000}{1.6} \\ \frac{2500}{1.6} & \frac{2300}{1.6} & \frac{2200}{1.6} \end{pmatrix}$$

By computing the fractions above, we obtain the desired result, expressed in euros:

$$\begin{array}{cc} & \begin{array}{ccc} q_1 & q_2 & q_3 \\ \downarrow & \downarrow & \downarrow \end{array} \\ \begin{array}{c} m_1 \rightarrow \\ m_2 \rightarrow \end{array} & \begin{pmatrix} 812.50 & 750.00 & 625.00 \\ 1562.50 & 1437.50 & 1375.00 \end{pmatrix} \end{array}$$

1.2.4. Matrix Product

Definition 5 ***(Product of a row vector and a column vector).*** Let $A = (a_1 \; \dots \; a_k)$ be a row vector and $B = \begin{pmatrix} b_1 \\ \vdots \\ b_k \end{pmatrix}$ be a column vector. The **product** $A \cdot B$ is the real number that is obtained by multiplying the elements at the same positions in each matrix, and then adding up the results of the multiplication.

$$(a_1 \; \dots \; a_k) \cdot \begin{pmatrix} b_1 \\ \vdots \\ b_k \end{pmatrix} = a_1 \cdot b_1 + \cdots + a_k \cdot b_k$$

Note that this product can only be made when the number of columns in matrix A coincides with the number of rows in matrix B.

Definition 6 ***(Product of two matrices).*** Given an $m \times k$ matrix, A, and a $k \times n$ matrix, B, the **product** $A \cdot B$ is a new $m \times n$ matrix where the element located in row "i" and column "j" is computed by multiplying row "i" of matrix A by column "j" of matrix B.

As we previously mentioned in the case of the product of a row vector and a column vector, note that the number of columns in the first matrix and the number of rows in the second one must coincide. Otherwise, it is not possible to multiply the matrices. Observe also, that the resulting matrix will have as many rows as the first matrix and as many columns as the second one.

$$\underset{m\times k}{A} \cdot \underset{k\times n}{B} = \underset{m\times n}{C}$$

Properties of the Product of Matrices

The product of matrices satisfies the following properties:

- Associativity: $(A \cdot B) \cdot C = A \cdot (B \cdot C)$
- Distributivity: $A \cdot (B + C) = A \cdot B + A \cdot C$
- The set of square matrices of dimension n has an identity element with respect to the product. This identity element is called the **identity matrix** of dimension n and is denoted by I_n. It is a diagonal matrix where the elements of the main diagonal are all equal to 1, that is

$$I_n = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

  and it holds that

$$A \cdot I_n = A = I_n \cdot A$$

- We should be aware that the product of matrices is in general not commutative. In other words, the order of the factors may change the result of the product.

Example 6. Given the matrices below

$$A = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} \quad B = \begin{pmatrix} 0 & -1 & 3 \\ 3 & 2 & 5 \end{pmatrix}$$

we wonder:

- If is it possible to calculate the products $A \cdot B$ and $B \cdot A$?
- What is the result of each of these products, in case both operations can be carried out?

A is a 2×2 matrix and B is a 2×3 matrix. Therefore, it is possible to compute the product $A \cdot B$

$$\underset{2\times 2}{A} \cdot \underset{2\times 3}{B} = \underset{2\times 3}{C}$$

but it is not possible to compute the product $B \cdot A$

$$\underset{2\times 3}{B} \cdot \underset{2\times 2}{A}$$

We compute $A \cdot B$

$$\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} \cdot \begin{pmatrix} 0 & -1 & 3 \\ 3 & 2 & 5 \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \end{pmatrix}$$

where

$$c_{11} = \begin{pmatrix} 1 & 3 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 3 \end{pmatrix} = 1 \cdot 0 + 3 \cdot 3 = 9$$

$$c_{12} = \begin{pmatrix} 1 & 3 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 2 \end{pmatrix} = 1 \cdot (-1) + 3 \cdot 2 = 5$$

$$c_{13} = \begin{pmatrix} 1 & 3 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 5 \end{pmatrix} = 1 \cdot 3 + 3 \cdot 5 = 18$$

$$c_{21} = \begin{pmatrix} 2 & 4 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 3 \end{pmatrix} = 2 \cdot 0 + 4 \cdot 3 = 12$$

$$c_{22} = \begin{pmatrix} 2 & 4 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 2 \end{pmatrix} = 2 \cdot (-1) + 4 \cdot 2 = 6$$

$$c_{23} = \begin{pmatrix} 2 & 4 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 5 \end{pmatrix} = 2 \cdot 3 + 4 \cdot 5 = 26$$

And we obtain $C = \begin{pmatrix} 9 & 5 & 18 \\ 12 & 6 & 26 \end{pmatrix}$.

Example 7. Consider the following method to compute the expenses of the purchases of several customers by using matrices. We build a matrix by writing the amounts of each commodity bought by a certain customer row-wise, using as many rows as customers. Next, we build a column vector that describes the prices of each of the commodities we consider. The product of these two matrices gives rise to a new column vector that depicts the total cost of the commodities purchased by each customer.

To keep it simple, suppose that there are two customers, A and B, and three commodities for sale, q_1, q_2 and q_3. The order of the first customer consists of 5 items of product q_1, 7 items of q_2 and 0 items of q_3. The second customer purchases 4, 0, and 8 items of commodities q_1, q_2 and q_3, respectively. If the price of the commodity q_1 is €230, the price of q_2 is €540 and the price of q_3 is €101, to find the expense of each customer we start by arranging the orders in a matrix.

$$\begin{array}{rc} & \begin{array}{ccc} q_1 & q_2 & q_3 \\ \downarrow & \downarrow & \downarrow \end{array} \\ \begin{array}{r} \text{customer A} \rightarrow \\ \text{customer B} \rightarrow \end{array} & \begin{pmatrix} 5 & 7 & 0 \\ 4 & 0 & 8 \end{pmatrix} \end{array}$$

Next, we build the price vector $\begin{pmatrix} 230 \\ 540 \\ 101 \end{pmatrix}$

and calculate the product:

$$\begin{pmatrix} 5 & 7 & 0 \\ 4 & 0 & 8 \end{pmatrix} \cdot \begin{pmatrix} 230 \\ 540 \\ 101 \end{pmatrix} = \begin{pmatrix} 5 \cdot 230 + 7 \cdot 540 + 0 \cdot 101 \\ 4 \cdot 230 + 0 \cdot 540 + 8 \cdot 101 \end{pmatrix} = \begin{pmatrix} 4930 \\ 1728 \end{pmatrix}$$

which means that customer A has spent €4930 and customer B has spent €1728.