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Quadratic Area Preserving Maps in $\mathbb{R}^2$

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Introduction

We consider a map $F : \mathbb{R}^2 \to \mathbb{R}^2$ and we are interested in the behaviour of the iterates of the points in $\mathbb{R}^2$ (or in some open connected subset of $\mathbb{R}^2$). If $z_0 = (x_0, y_0) \in \mathbb{R}^2$ is some initial point, we want to see what is the behaviour of the iterates $z_{k+1} = F(z_k)$. The set $\{z_k, k \in \mathbb{N}\}$ is known as the orbit of z_0.

There are different ways to produce such a map, F. We can give it explicitly or we can consider an autonomous ordinary differential equation (ODE) $dz/dt = f(z)$ and define $F(z)$ as the image of a point z under the flow of the ODE after a fixed time τ: $F(z) = \varphi(\tau, z)$ or simply $\varphi_\tau(z)$, the so-called *time-τ map*.

Another interesting way to arrive at F is to use the so-called *Poincaré map*. Assume we have an ODE, $dw/dt = f(w)$, in $\mathbb{R}^3$ (or in some open connected subset of it or in a 3D manifold) and let Σ be a surface in $\mathbb{R}^3$ given, for instance, by $g(w) = 0$ assuming g to be a smooth function. If the flow of the ODE intersects Σ transversally (that is, $f(w)$ is not tangent to Σ at any of the points $w \in \Sigma$) we can do the following. Take an initial point, $w \in \Sigma$, and look for the solution $\varphi(t, w)$ of the ODE starting at w. If there exists some $t = t(w) > 0$ such that $\varphi(\tau(w), w) \in \Sigma$ while $\varphi(t, w) \notin \Sigma$ for $0 < t < t(w)$, we define this point as the image $\mathcal{P}(w)$ of w under the Poincaré map $\mathcal{P}$. This is a map defined on the 2D manifold Σ.

A very important case arises when the vector field f comes from a Hamiltonian with two degrees of freedom on a given level of the energy. A *Hamiltonian system* of equations with n degrees of freedom and Hamiltonian $H(q, p)$ has equations of the form

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}, \quad i = 1, \ldots, n \tag{1}$$

for $(q, p) \in \mathbb{R}^n \times \mathbb{R}^n$ (or in suitable manifolds) and it is immediately clear that $H(q, p)$ is preserved along the solutions of the ODE. The value of $H(q, p)$ is known as *the energy*. If $n = 2$ and we consider a given value of the energy, we have an ODE in 3D. If we can define a Poincaré map, as we did before, we again obtain a map on a 2D manifold, Σ.

In that case, the map $\mathcal{P}$ has an important property: *it preserves the area in* Σ. Hence, from now on we can simplify the presentation. We will consider $\mathbb{R}^2$ instead of a general surface Σ, we will denote the map simply as F and, furthermore, we will assume that it is area preserving. This implies that one has $\det(DF(z)) = 1$ for all $z \in \mathbb{R}^2$, where DF denotes the differential of F. For shortness we will denote Area Preserving Maps as *APM*.

The simplest case concerns maps F of degree 1. They are of the form $F(z) = Mz + b, b \in \mathbb{R}^2$, where M is a 2×2 matrix. If we skip the trivial cases with double eigenvalue equal to 1, it is always possible to assume $b = 0$ by shifting the origin. The eigenvalues of M can be of the form

1) $\exp(\pm \mathrm{i}\alpha),\ \alpha \in (0, \pi)$, a case known as *elliptic*,
2) $\lambda, \lambda^{-1},\ \lambda > 1$, a case known as *hyperbolic*,
3) 1 double, but M cannot be diagonalised, a case known as *shear*.

Two further cases can also appear with eigenvalues $\lambda, \lambda^{-1},\ \lambda < -1$ (usually known as hyperbolic with reflection) and -1 double, again with M non-diagonalisable. They reduce to 2) and 3) above by composing with the central symmetry $-\mathrm{Id}$ or taking F^2 instead of F.

For the three cases above, after a change of coordinates, the matrix M can be reduced to one of the types:

$$\begin{pmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{pmatrix}, \qquad \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \qquad \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}. \tag{2}$$

In the elliptic case, the dynamics is essentially a rotation according to (2). All the points have bounded orbits, confined to ellipses. We note also that if α is of the form $\alpha = \frac{m}{n}2\pi,\ \frac{m}{n} \in \mathbb{Q},\ (m, n) = 1$, then all the points are fixed under F^n, an extremely degenerate situation. That is, all the points are periodic with minimal period n. We recall that a point z is called *periodic with minimal period* $n > 1$ if $F^n(z) = z$ and $F^k(z) \neq z$ for $k = 1, 2, \ldots, n-1$.

In the hyperbolic case, the dynamics is of saddle type. Written as $F(x, y) = (\lambda x, \lambda^{-1}y)$, according to (2), all the points with $x_0 \neq 0$ escape to infinity, while the points with $x_0 = 0$ tend to the origin.

In the shear case, we can write the map as $F(x, y) = (x + y, y)$, as given by (2). All points with $y_0 \neq 0$ tend to infinity, while the points with $y_0 = 0$ are all fixed.

Figure 1 shows the three cases, with an initial point, labelled as 0, and the images labelled as 1, 2, 3, ... It is very important to note that in this

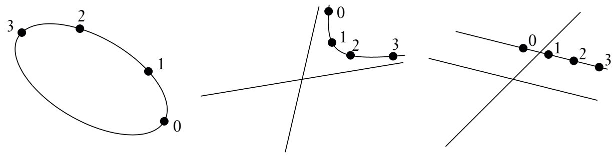

Figure 1. Examples of the dynamics of the three models of linear maps. From left to right: rotation, saddle and shear. The normal forms of the corresponding matrices are given in (2).

linear case *all the iterates belong to curves* which are solutions of linear ODE which preserve the area. That is, ODE associated with a Hamiltonian which is quadratic. In fact there exists a Hamiltonian such that the map coincides with the time-1 map of the Hamiltonian flow.

If for a general map F in $\mathbb{R}^2$ there exists a non-constant function G which is preserved by F, i.e., $G(F(z)) = G(z)$, we say that the map is *integrable*.

Hence, we pass to more interesting cases. The first that comes to mind is the case of *quadratic maps*. If we define $F(x,y) = (F_1(x,y), F_2(x,y))$ then F_1 and F_2 are quadratic polynomials in (x,y). This case is extremely relevant because

a) It allows for a reduction which decreases in a dramatic way the number of parameters involved in the map. Furthermore it has some relevant symmetries and a very simple geometric interpretation. This will be considered in Section 2.

b) It appears in a natural way as a very good approximation in some parts of $\mathbb{R}^2$ when we consider arbitrary APM. In particular when we study Poincaré maps of a Hamiltonian with 2 degrees of freedom.

c) It is a paradigmatic model. Many problems concerning: the existence of invariant curves diffeomorphic to $\mathbb{S}^1$ (Section 4); the role of invariant manifolds of hyperbolic fixed or periodic points and how they lead to the existence of chaos (Section 5); the geometrical mechanisms leading to the destruction of invariant curves (Section 6); and quantitative measures of different properties (Sections 7 and 8) for general APM [26], can all be understood thanks to our knowledge of the quadratic case.

Several questions still remain open, a few of which will be presented in Section 9.

Furthermore quadratic APM are *analytic*, a fact which allows us to apply useful results for analytic maps, and even more: they are *entire maps*. For a complete discussion of properties of different kinds of Hénon map and some applications, see [19]. Several of the figures and ideas presented here are taken from this paper.

Reduction and symmetries

At the end of the sixties, *Michel Hénon* [13] started the study of quadratic APM. Later on, on the middle of the seventies, he looked at the dissipative case [14]. In that case the Jacobian is constant but of absolute value below 1. It gives rise to the popular *Hénon attractor*.

However a remarkable fact, that is not difficult to prove, is that a constant Jacobian plus suitable shifts of the origin and scaling of variables, allow us to write any quadratic map in $\mathbb{R}^2$ in the form:

$$F : (x, y) \to (1 - ax^2 + y, bx) \tag{3}$$

for some constants a, b. When a tends to 0, scaling should be performed in a different way. Obviously the Jacobian is equal to $-b$ and, hence, the conservative case is obtained for $b = -1$.

The conservative map has a very simple geometric interpretation. It is the composition of two maps. The first one is $(x, y) \to (x, y + 1 - ax^2)$, one of the so-called *de Jonquières* maps, while the second is just a rotation by an angle of $-\pi/2$. Figure 2 shows, for $a = -1/2$, the square $[-3, 3]^2$ (in red), the first image (in green) and part of the next two images (in blue and magenta, respectively). One can ask whether all points will escape for future iterations. As an answer to this question, we plot in black the set of points which remain bounded for all iterations and the selected value of a. This is the kind of objects we want to study together with how they change as a function of the parameter.

A better representation of the quadratic APM, to be used in what follows, is:

$$F_c \begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} x + 2y + \frac{c}{2}(1 - (x+y)^2) \\ y + \frac{c}{2}(1 - (x+y)^2) \end{pmatrix}, \tag{4}$$

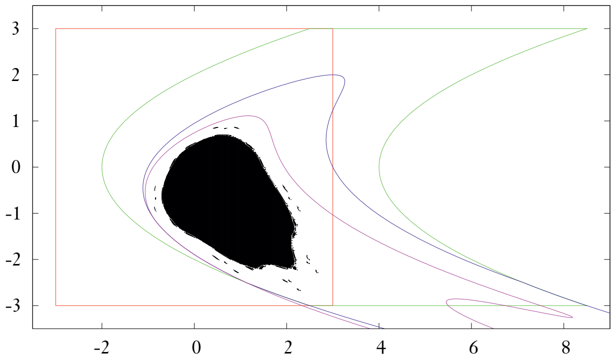

Figure 2. The square $[-3,3]^2$ (in red) and the first three images of it under the Hénon map with $a = -0.5, b = -1$, shown in green, blue and magenta, respectively. The last two have parts outside the frame shown here. The set in black displays the invariant set of points which remain bounded under all iterations.

where it is enough to consider $c > 0$. It is obtained from a minimal modification of the version given in [34]. The subscript c in F_c is introduced to stress that the map depends on this parameter. The map has two fixed points. One of them is located at $(-1,0)$ and it is hyperbolic for all $c > 0$. The other one, located at $(1,0)$, is elliptic for $0 < c < 2$, parabolic (concretely of shear type) for $c = 2$ and hyperbolic with reflection for $c > 2$.

It can immediately be checked that the inverse map can be expressed as $F_c^{-1} = SF_cS$, where S is the symmetry given by $S(x,y) = (x,-y)$. Hence, if we define $R = SF_c$, clearly an involution like S, we have $F_c = SR$ and $F_c^{-1} = RS$. Both S and R are called *reversors*. We can consider the sets of fixed points of both reversors, i.e., either points $z = (x,y)$ such that $S(z) = z$ (which are the points with $y = 0$) or points such that $R(z) = z$, which belong to a parabola.

There is an important property due to the existence of reversors of a map F. If K is a reversor and z^* is a point in the set $\mathrm{Fix}(K)$ of fixed points of K, then, if there exists $m \in \mathbb{N}$ such that $F^m(z^*) \in \mathrm{Fix}(K)$, the point z^* is periodic under F. Obviously, if $F^m(z^*) = z^*$, it has period m; and if $F^m(z^*) \neq z^*$, it has period $2m$. It is also clear that a fixed point can always be considered as periodic with period 1.

To study the local linear properties at a periodic point z^* with period m, it is enough to consider the linear map defined by $DF^m(z^*)$ and to use the description of the dynamics in Section 1. Accordingly, the point will be called elliptic, hyperbolic, hyperbolic with reflection or, if the eigenvalues of $DF^m(z^*)$ are $+1$ double or -1 double, parabolic.

A limit flow: comparison with the maps

A useful device for a preliminary study of the dynamics of a map F, provided it is *close to the identity map*, is to look for the existence of some ODE such that the time-1 map associated with the flow gives a good approximation to F. Looking at (4) it suggests to introduce new variables $(\xi, \eta) = (x, 2y/\sqrt{c})$. Then, in the (ξ, η) variables, F_c differs from Id by $\mathcal{O}(\sqrt{c})$. A scaling of time also by $\sqrt{c}$ leads to:

$$\frac{d\xi}{dt} = \eta, \quad \frac{d\eta}{dt} = 1 - \xi^2, \tag{5}$$

an ODE which is Hamiltonian with $H(\xi, \eta) = \frac{1}{2}\eta^2 - \xi + \frac{1}{3}\xi^3$ (see (1)). So, the solutions are contained in the level curves of H. The dynamics of (5) is elementary and the main features are shown in Figure 3 left. It has also $(-1, 0)$ and $(1, 0)$ as fixed points, of hyperbolic and elliptic type respectively.

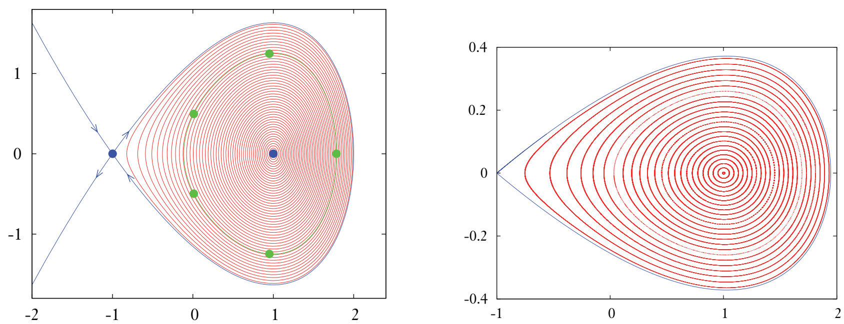

Figure 3. Left: the phase portrait of system (5). Fixed points are shown in blue, as are the invariant manifolds of the hyperbolic point. The periodic orbits are shown in red. See the text for details about the periodic orbit and the points shown in green. Right: the right branches of the invariant manifolds of the hyperbolic point and (part of) the orbits of several initial points under F_c, for $c = 0.2$.