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Differences of weighted composition operators

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Abstract

We consider differences of weighted composition operators between given weighted Bergman spaces H_v^∞ of infinite order and characterize boundedness and the essential norm of these differences.

I. Introduction

Let v, w be strictly positive bounded continuous functions (*weights*) on the open unit disk D of the complex plane $\mathbb{C}$. We define the weighted Bergman space of infinite order as follows

$$H_v^\infty := \{f \in H(D);\ \|f\|_v := \sup_{z\in D} v(z)|f(z)| < \infty\},$$

where $H(D)$ denotes the space of all holomorphic functions. Endowed with norm $\|.\|_v$, the space H_v^∞ is a Banach space. Such spaces have been studied by various authors while investigating growth conditions of analytic functions. As an assertment of papers on this topic we would like to mention [18, 20, 21, 11, 2, 13, 14, 9, 3].

Furthermore we consider analytic self maps ϕ_1, ϕ_2 of D as well as analytic functions $\psi_1, \psi_2 : D \to \mathbb{C}$. These maps induce weighted composition operators

$$C_{\phi_i,\psi_i} : H(D) \to H(D),\quad f \to \psi_i(f \circ \phi_i),\quad i = 1, 2\,.$$

Composition operators and weighted composition operators have been investigated on various spaces and by several authors, see e.g. [5, 4, 7, 15, 16, 17, 6, 12]. We are interested in differences $C_{\phi_1,\psi_1} - C_{\phi_2,\psi_2}$ of weighted composition operators acting

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between weighted Bergman spaces of infinite order. In [12] we studied the essential norm of such differences in case of quite special weights. The aim of this article is to characterize boundedness and the essential norm of differences $C_{\phi_1,\psi_1} - C_{\phi_2,\psi_2}$ under more general assumptions on the weights. The estimates we obtain in this note differ from the conditions given in [12]. Otherwise for $\psi_1 = \psi_2 = 1$ we get exactly the result of [6].

II. Notations, definitions and auxiliary results

For notation and general information on composition operators see the excellent monographs [8, 19]. Let us denote the closed unit ball of H_v^∞ by B_v^∞. When dealing with weighted Bergman spaces of infinite order an important tool is the so called *associated weight* introduced by Anderson and Duncan in [1] and thoroughly studied by Bierstedt, Bonet and Taskinen in [3]. For a weight v the associated weight $\tilde{v}$ is given by

$$\tilde{v}(z) := \frac{1}{\sup\{|f(z)|;\, f \in B_v^\infty\}} = \frac{1}{\|\delta_z\|_{{H_v^\infty}'}}, \quad z \in D,$$

where δ_z denotes the point evaluation of z. By [3] the associated weights have the following properties:

(i) $\tilde{v}$ is continuous,
(ii) $\tilde{v} \geq v > 0$,
(iii) for every $z \in D$ we can find $f_z \in B_v^\infty$ such that $f_z(z) = \frac{1}{\tilde{v}(z)}$.

A weight v is called *essential* if there is a constant $C > 0$ such that

$$v(z) \leq \tilde{v}(z) \leq Cv(z) \text{ for every } z \in D.$$

Examples of essential weights as well as conditions when weights are essential may be found in [3, 4, 5]. We are especially interested in *radial* weights, i.e. weights which satisfy $v(z) = v(|z|)$ for every $z \in D$. If a radial weight v satisfies the Lusky condition $(L1)$ (due to Lusky [13])

$$(L1) \quad \inf_k \frac{v(1-2^{-k-1})}{v(1-2^{-k})} > 0,$$

then v is essential.

We say that a weight v is *typical*, if it is radial, non-increasing with respect to $|z|$ and $\lim_{|z|\to 1-} v(z) = 0$. In the sequel every radial weight is assumed to be non-increasing. In order to treat differences of composition operators we need some geometric data. Recall that for any $z \in D$, φ_z is the Möbius transformation of D which interchanges the origin and z, namely, $\varphi_z(w) = \frac{z-w}{1-\bar{z}w}$, $w \in D$. The *pseudohyperbolic distance* $\rho(z,w)$ for $z, w \in D$ is defined by $\rho(z,w) = |\varphi_z(w)| = \left|\frac{z-w}{1-\bar{z}w}\right|$.

An operator $T \in L(E,F)$ from a Banach space E to the Banach space F is called *compact* if it maps the closed unit ball of E onto a relatively compact set in F. The essential norm of a continuous linear operator T is defined by $\|T\|_e := \inf\{\|T-K\|;\ K \text{ is compact}\}$, i.e. the essential norm is the distance to the compact operators. Finally, let us list up some auxiliary results we need for proving our results.

Lemma 1 (Bonet, Lindström, Wolf, [6])

Let v be a radial weight satisfying the Lusky condition $(L1)$ and let $f \in H_v^\infty$. Then there exists a constant C_v (depending on the weight v) such that

$$|f(z) - f(p)| \leq C_v ||f||_v \max\left\{\frac{1}{v(z)}, \frac{1}{v(p)}\right\} \rho(z,p)$$

for all $z, p \in D$.

Theorem 2 (Contreras, Hernández-Díaz, [7])

Let v and w be weights and $\phi : D \to D$ and $\psi : D \to \mathbb{C}$ be analytic. Then the operator $C_{\phi,\psi} : H_v^\infty \to H_w^\infty$ is bounded if and only if $\sup_{z\in D} |\psi(z)| \frac{w(z)}{\tilde{v}(\phi(z))} < \infty$.

III. Continuity and essential norm of differences of weighted composition operators

First, recall that the operator $C_{\phi_1,\psi_1} - C_{\phi_2,\psi_2} : H_v^\infty \to H_w^\infty$ is bounded if and only if

$$\sup_{z\in D} w(z) \sup\left\{|\psi_1(z) f(\phi_1(z)) - \psi_2(z) f(\phi_2(z))|;\ f \in B_v^\infty\right\} < \infty\,.$$

Proposition 3

Let v and w be weights such that v is radial and satisfies $(L1)$. Then $C_{\phi_1,\psi_1} - C_{\phi_2,\psi_2} : H_v^\infty \to H_w^\infty$ is bounded if and only if

(i) $\sup_{z\in D}\, w(z)\, \min\left\{|\psi_1(z)|, |\psi_2(z)|\right\}\, \rho\big(\phi_1(z), \phi_2(z)\big)\, \max\left\{\frac{1}{\tilde{v}(\phi_1(z)}, \frac{1}{\tilde{v}(\phi_2(z))}\right\} < \infty\,,$

(ii) $\sup_{z\in D}\, |\psi_1(z) - \psi_2(z)|\, w(z)\, \max\left\{\frac{1}{\tilde{v}(\phi_1(z))}, \frac{1}{\tilde{v}(\phi_2(z))}\right\} < \infty\,.$

Proof. We suppose that the operator $C_{\phi_1} - C_{\phi_2}$ is bounded, i.e.

$$M = \sup_{z\in D} w(z) \sup\left\{|\psi_1(z) f(\phi_1(z)) - \psi_2(z) f(\phi_2(z))|;\ f \in B_v^\infty\right\} < \infty\,.$$

Let us start with proving condition (i) indirectly. W.l.o.g. we can find a sequence $(z_n)_{n\in\mathbb{N}} \subset D$ such that

$$w(z_n) \min\left\{|\psi_1(z_n), |\psi_2(z_n)|\right\} \rho\big(\phi_1(z_n), \phi_2(z_n)\big)\, \max\left\{\frac{1}{\tilde{v}(\phi_1(z_n))}, \frac{1}{\tilde{v}(\phi_2(z_n))}\right\} \geq n\,.$$

We fix $n \in \mathbb{N}$ and distinguish the following cases:

First, we assume $\max\left\{\frac{1}{\tilde{v}(\phi_1(z_n))}, \frac{1}{\tilde{v}(\phi_2(z_n))}\right\} = \frac{1}{\tilde{v}(\phi_1(z_n))}$. Then there is a function $f_n \in B_v^\infty$ such that $|f_n(\phi_1(z_n))| = \frac{1}{\tilde{v}(\phi_1(z_n))}$. Next, we put $h_n(z) := f_n(z)\varphi_{\phi_2(z_n)}(z)$ for every $z \in D$. Obviously h_n belongs to B_v^∞. This yields

$$\begin{aligned}
M \geq\ & w(z_n)\, |\psi_1(z_n)\, h_n(\phi_1(z_n)) - \psi_2(z_n)\, h_n(\phi_2(z_n))| \\
\geq\ & w(z_n)\, |\psi_1(z_n)|\, |f_n(\phi_1(z_n))|\, \rho(\phi_1(z_n), \phi_2(z_n)) \\
\geq\ & w(z_n)\, \min\left\{|\psi_1(z_n)|\,,\, |\psi_2(z_n)|\right\} \left|\frac{1}{\tilde{v}(\phi_1(z_n))}\right| \rho\big(\phi_1(z_n),\, \phi_2(z_n)\big) \\
\geq\ & n\,.
\end{aligned}$$

Secondly, we suppose $\max\left\{\frac{1}{\tilde{v}(\phi_1(z_n))}, \frac{1}{\tilde{v}(\phi_2(z_n))}\right\} = \frac{1}{\tilde{v}(\phi_2(z_n))}$. Choosing $f_n \in B_v^\infty$ such that $|f_n(\phi_2(z_n))| = \frac{1}{\tilde{v}(\phi_2(z_n))}$ we now set $h_n(z) := f_n(z)\varphi_{\phi_1(z_n))}(z)$ for every $z \in D$ and get analogously to the previous case

$$M \geq w(z_n) \min\{|\psi_1(z_n)|\,,\, |\psi_2(z_n)|\} \left|\frac{1}{\tilde{v}(\phi_2(z_n))}\right| \rho(\phi_1(z_n),\ \phi_2(z_n)) \geq n\,.$$

Joining both cases, for every $n \in \mathbb{N}$ we obtain

$$\begin{aligned} M \geq\ & w(z_n)\ \min\{|\psi_1(z_n)|\,,\, |\psi_2(z_n)|\}\, \rho(\phi_1(z_n), \phi_2(z_n)) \max\left\{\frac{1}{\tilde{v}(\phi_1(z_n)}\,,\, \frac{1}{\tilde{v}(\phi_2(z_n))}\right\} \\ \geq\ & n, \end{aligned}$$

which is a contradiction.

It remains to show (ii). Fix $z \in D$ and consider the following cases:

First, we suppose $\min\{|\psi_1(z)|\,,\ |\psi_2(z)|\} = |\psi_1(z)|$ and select $f_z \in B_v^\infty$ such that $|f_z(\phi_2(z))|\,\tilde{v}(\phi_2(z)) = 1$. Hence an application of Lemma 1 gives

$$\begin{aligned} w(z)\,|\psi_1(z) - \psi_2(z)|\,\frac{1}{\tilde{v}(\phi_2(z))} =\ & w(z)\,|\psi_2(z) - \psi_1(z)|\,|f_z(\phi_2(z))| \\ \leq\ & w(z)\,|\psi_2(z) f_z(\phi_2(z)) - \psi_1(z) f_z(\phi_1(z))| \\ & +\ w(z)\,|\psi_1(z) f_z(\phi_1(z)) - \psi_1(z) f_z(\phi_2(z))| \\ \leq\ & M + w(z)\,|\psi_1(z)|\,|f_z(\phi_2(z)) - f_z(\phi_1(z))| \\ \leq\ & M + C_v w(z)\ \min\{|\psi_1(z)|, |\psi_2(z)|\}\, \rho(\phi_1(z), \phi_2(z)) \\ & \times\ \max\left\{\frac{1}{\tilde{v}(\phi_1(z))}, \frac{1}{\tilde{v}(\phi_2(z))}\right\}. \end{aligned}$$

Secondly, let $\min\{|\psi_1(z)|, |\psi_2(z)|\} = |\psi_2(z)|$. Choose $f_z \in B_v^\infty$ with $|f_z(\phi_1(z))|\,\tilde{v}(\phi_1(z)) = 1$. Proceeding as in the previous case we get

$$\begin{aligned} w(z)\,|\psi_1(z) - \psi_2(z)|\ \frac{1}{\tilde{v}(\phi_1(z))} \leq\ & M + C_v w(z) \min\ \{|\psi_1(z)|\,,\ |\psi_2(z)|\}\, \rho(\phi_1(z), \phi_2(z)) \\ & \times\ \max\left\{\frac{1}{\tilde{v}(\phi_1(z))}\,,\, \frac{1}{\tilde{v}(\phi_2(z))}\right\}. \end{aligned}$$

Joining both cases and using (i), we can deduce condition (ii).

For the converse fix $f \in B_v^\infty$ and $z \in D$ and distinguish the following cases:

If $\min\{|\psi_1(z)|\,,\ |\psi_2(z)|\} = |\psi_1(z)|$, using Lemma 1 yields

$$\begin{aligned}
|\psi_1(z)f(\phi_1(z)) - \psi_2(z)f(\phi_2(z))| \leq\ & |\psi_1(z)|\,|f(\phi_1(z)) - f(\phi_2(z))| \\
& + |\psi_1(z) - \psi_2(z)|\,|f(\phi_2(z))| \\
\leq\ & C_v\,|\psi_1(z)|\,\rho(\phi_1(z), \phi_2(z)) \\
& \times \max\left\{\frac{1}{v(\phi_1(z))}, \frac{1}{v(\phi_2(z))}\right\} \\
& + |\psi_1(z) - \psi_2(z)|\ \frac{1}{v(\phi_2(z))} \\
\leq\ & C_v \min\{|\psi_1(z)|,\ |\psi_2(z)|\}\,\rho(\phi_1(z), \phi_2(z)) \\
& \times \max\left\{\frac{1}{v(\phi_1(z))}, \frac{1}{v(\phi_2(z))}\right\} \\
& + |\psi_1(z) - \psi_2(z)| \max\left\{\frac{1}{v(\phi_1(z))}, \frac{1}{v(\phi_2(z))}\right\}.
\end{aligned}$$

In case $\min\{|\psi_1(z)|, |\psi_2(z)|\} = |\psi_2(z)|$ we proceed analogously to get

$$\sup_{z\in D} |\psi_1(z)f(\phi_1(z)) - \psi_2(z)f(\phi_2(z))| < \infty\,.$$

From this we conclude that the operator is bounded if the above conditions are satisfied. $\square$

Next, we give an example of non-bounded operators C_{ϕ_1,ψ_1} and C_{ϕ_2,ψ_2} such that the difference is bounded.

Example 4 Choose $w(z) = v(z) = 1 - |z| = \tilde{v}(z)$, $\phi_1(z) = \frac{z+1}{2}$ and $\phi_2(z) = \frac{z+1}{2} + t(z-1)^3$ for every $z \in D$ such that t is real and $|t|$ so small that ϕ_2 maps D into D as well as $\psi_1(z) = \psi_2(z) = \frac{1}{1-z}$ for every $z \in D$. Obviously v and w are radial weights and v has condition $(L1)$. Moreover the functions ψ_1 and ψ_2 belong to the space H_w^∞. By [7] Proposition 3.1 C_{ϕ_1,ψ_1} is not bounded since for $z = r \in \mathbb{R}$ we have $|\psi_1(r)|\frac{w(r)}{\tilde{v}(\phi_1(r))} = \frac{2}{1-r} \to \infty$ if $r \to 1$. Analogously we can show that C_{ϕ_2,ψ_2} is not bounded. By [15] Example 1 we know $\rho(\phi_1(z), \phi_2(z)) \leq \frac{|t|}{\delta}|z-1| < \infty$, where δ is a constant. This yields

$$\sup_{z\in D} |\psi_1(z)|\frac{w(z)}{v(\phi_1(z))}\rho(\phi_1(z), \phi_2(z)) \leq \sup_{z\in D} \frac{1}{|1-z|}\frac{1-|z|}{1-|\frac{z+1}{2}|}\frac{|t|}{\delta}|z-1| < \infty$$

and $\displaystyle\sup_{z\in D} |\psi_1(z) - \psi_2(z)|\frac{w(z)}{v(\phi_1(z))} = 0$ as well as

$$\sup_{z\in D} |\psi_1(z)|\frac{w(z)}{\tilde{v}(\phi_2(z))}\rho(\phi_1(z), \phi_2(z)) \leq \sup_{z\in D} \frac{1}{|1-z|}\frac{1-|z|}{1-|\frac{z+1}{2} + t(z-1)^3|}\frac{|t|}{\delta}|z-1| < \infty$$

and $\displaystyle\sup_{z\in D} |\psi_1(z) - \psi_2(z)|\frac{w(z)}{v(\phi_2(z))} = 0.$

Hence the corresponding difference $C_{\phi_1,\psi_1} - C_{\phi_2,\psi_2}$ is bounded.

Theorem 5

Let v and w be radial weights such that v is typical and satisfies the Lusky condition $(L1)$. Let $\psi_1, \psi_2 \in H_w^\infty$ such that there are constants $\alpha, \beta > 0$ with $\alpha \leq |\frac{\psi_1(z)}{\psi_2(z)}| \leq \beta$ for every $z \in D$. If $\phi_1, \phi_2 : D \to D$ are analytic maps such that $||\phi_1||_\infty = ||\phi_2||_\infty = 1$ and $C_{\phi_1,\psi_1}, C_{\phi_2,\psi_2} : H_v^\infty \to H_w^\infty$ both are bounded, then the essential norm $\|C_{\phi_1,\psi_1} - C_{\phi_2,\psi_2}\|_e$ is equivalent to the maximum of the following expressions:

(i) $\limsup_{|\phi_1(z)|\to 1-} |\psi_1(z)|\, \rho(\phi_1(z), \phi_2(z)) \max\left\{\frac{1}{\tilde{v}(\phi_1(z))}, \frac{1}{\tilde{v}(\phi_2(z))}\right\}$,

(ii) $\limsup_{|\phi_2(z)|\to 1-} |\psi_2(z)|\, \rho(\phi_1(z), \phi_2(z)) \max\left\{\frac{1}{\tilde{v}(\phi_1(z))}, \frac{1}{\tilde{v}(\phi_2(z))}\right\}$,

(iii) $\limsup_{\min\{|\phi_1(z)|,|\phi_2(z)|\}\to 1-} w(z)|\psi_1(z) - \psi_2(z)| \max\left\{\frac{1}{\tilde{v}(\phi_1(z))}, \frac{1}{\tilde{v}(\phi_2(z))}\right\}$.

Proof. First we want to prove that there is a constant $C > 0$ such that $C \max\{(i), (ii), (iii)\} \leq \|C_{\phi_1,\psi_1} - C_{\phi_2,\psi_2}\|_e$. Let us start with considering (i).

(i) Let $(z_n)_n \in D$ be a sequence with $|\phi_1(z_n)| \to 1$ such that

$$\lim_n |\psi_1(z_n)|\, w(z_n) \max\left\{\frac{1}{\tilde{v}(\phi_1(z_n))}, \frac{1}{\tilde{v}(\phi_2(z_n))}\right\} \rho(\phi_1(z_n), \phi_2(z_n))$$
$$= \limsup_{|\phi_1(z)|\to 1-} w(z)|\psi_1(z)|\, \rho(\phi_1(z), \phi_2(z)) \max\left\{\frac{1}{\tilde{v}(\phi_1(z))}, \frac{1}{\tilde{v}(\phi_2(z))}\right\}.$$

In case $\max\left\{\frac{1}{\tilde{v}(\phi_1(z_n))}, \frac{1}{\tilde{v}(\phi_2(z_n))}\right\} = \frac{1}{\tilde{v}(\phi_1(z_n))}$, since $|\phi_1(z_n)| \to 1$, by going to a subsequence if necessary, we can use the proof of Theorem 3.1 in [10] to find functions $(g_n)_n \in H^\infty$ such that

$$\sum_{n=1}^{\infty} |g_n(z)| \leq 1 \quad \text{for all} \quad z \in D,$$

and $g_n(\phi_1(z_n)) > 1 - (\frac{1}{2})^n$ for every n. Then $\lim_n |g_n(\phi_1(z_n))| = 1$. In the sequel we want to explain roughly how to construct these functions following the proof of Theorem 3.1 in [10].

First we put $k(z) := \frac{z+1}{2}$ for every $z \in D$. Then k is holomorphic on D and continuous on $\overline{D}$ such that $k(1) = 1$ and $|k| < 1$ on $\overline{D}\backslash\{1\}$. Next we consider $u(z) := \frac{z-1}{2}$ as well as $u_n(z) := u(z)^{\frac{1}{n}}$ for every $z \in D$. Each u_n is holomorphic on D and continuous on $\overline{D}$ such that $\|u_n\|_\infty = 1$, $u_n(1) = 0$ and $|u_n(z)| \to 1$ for every $z \in D$. By induction we can find two increasing sequences $(m_l)_l$, $(j_l)_l \subset \mathbb{N}$, a sequence $(c_l)_l$ of complex numbers with $|c_l| < 1$ for every $l \in \mathbb{N}$ and a subsequence $(\phi(z_l))_l$ of $(\phi(z_n))_n$ such that

$$\sup_{z\in\overline{D}} \sum_{l=1}^{N} |(c_l k^{m_l} u_{j_l})(z)| < 1 \quad \text{for every} \quad N \in \mathbb{N} \text{ and}$$
$$c_N(k^{m_N} u_{j_N})(\phi(z_N)) > 1 - \frac{1}{2^N} \quad \text{for every} \quad N \in \mathbb{N}.$$

Putting $g_N(z) := (c_N k^{m_N} u_{j_N})(z)$ for every $z \in D$ and for every $N \in \mathbb{N}$, we obtain the functions above. For more details we refer the reader to [10].

For every n we can also find $f_n \in B_v^\infty$ with $f_n(\phi_1(z_n)) = \frac{1}{\tilde{v}(\phi_1(z_n))}$. Set $h_n(z) := g_n(z)\ \varphi_{\phi_2(z_n)}(z)\ f_n(z)$. Thus, $h_n \in H_v^\infty$ with $||h_n||_v \le 1$. Since the standard basis $(e_n)_n$ for c_0 tends weakly to zero, so does $(h_n)_n$. Now let $K : H_v^\infty \to H_w^\infty$ be a compact operator. Then $\lim_{n\to\infty} ||Kh_n||_w = 0$. For each n,

$$||\psi_1 C_{\phi_1} - \psi_2 C_{\phi_2} - K|| \ge ||(\psi_1 C_{\phi_1} - \psi_2 C_{\phi_2})h_n||_w - ||Kh_n||_w,$$

and thus we conclude that

$$\begin{aligned}
||\psi_1 C_{\phi_1} - \psi_2 C_{\phi_2} - K|| &\ge \limsup_n ||\psi_1(h_n \circ \phi_1) - \psi_2(h_n \circ \phi_2)||_w \\
&\ge \limsup_n w(z_n)|\psi_1(z_n)h_n(\phi_1(z_n)) - \psi_2(z_n)h_n(\phi_2(z_n))| \\
&= \limsup_n w(z_n)|\psi_1(z_n)||g_n(\phi_1(z_n))||\varphi_{\phi_2(z_n)}(\phi_1(z_n))\ f_n(\phi_1(z_n))| \\
&= \limsup_n |\psi_1(z_n)| \frac{w(z_n)}{\tilde{v}(\phi_1(z_n))} \rho(\phi_2(z_n), \phi_1(z_n))
\end{aligned}$$

Now, let us assume $\max\left\{\frac{1}{\tilde{v}(\phi_1(z_n))}, \frac{1}{\tilde{v}(\phi_2(z_n))}\right\} = \frac{1}{\tilde{v}(\phi_2(z_n))}$. If $|\phi_2(z_n)| \not\to 1$ there are only finitely many n for which this is the case, since v is typical und $|\phi_1(z_n)| \to 1$. Then these $n's$ can be omitted and we have the first case. If $|\phi_2(z_n)| \to 1$, analogously to the previous case, choose functions $(g_n)_n \in H^\infty$ with

$$\sum_{n=1}^{\infty} |g_n(z)| \le 1 \quad \text{for all} \quad z \in D,$$

and $\lim_n |g_n(\phi_2(z_n))| = 1$. For every n we select $f_n \in B_v^\infty$ such that $f_n(\phi_2(z_n)) = \frac{1}{\tilde{v}(\phi_2(z_n))}$ and set $h_n(z) := g_n(z)\ \varphi_{\phi_1(z_n)}(z)\ f_n(z)$. Proceeding in the same way as above and taking into account that by assumption $\alpha \le \frac{|\psi_1(z_n)|}{|\psi_2(z_n)|} \le \beta$ for every $n \in \mathbb{N}$ we get

$$\begin{aligned}
\|\psi_1 C_{\phi_1} - \psi_2 C_{\phi_2} - K\| &\ge \limsup_n |\psi_2(z_n)| \frac{w(z_n)}{\tilde{v}(\phi_2(z_n))} \rho(\phi_1(z_n), \phi_2(z_n)) \\
&\ge \frac{1}{\beta} \limsup_n |\psi_2(z_n)| \frac{|\psi_1(z_n)|}{|\psi_2(z_n)|} \frac{w(z_n)}{\tilde{v}(\phi_1(z_n))} \rho(\phi_1(z_n), \phi_2(z_n)) \\
&= \frac{1}{\beta} \limsup_n |\psi_1(z_n)| \frac{w(z_n)}{\tilde{v}(\phi_2(z_n))} \rho(\phi_1(z_n), \phi_2(z_n)) \, .
\end{aligned}$$

Joining both cases we get (i).

(ii) follows in an analogous way.

(iii) If $\rho(\phi_1(z), \phi_2(z)) \to \sigma \neq 0$, when $|\phi_1(z)| \to 1$ and $|\phi_2(z)| \to 1$, then (iii) follows from (i) and (ii). Therefore we can assume that $\rho(\phi_1(z), \phi_2(z)) \to 0$ if $|\phi_1(z)| \to 1$ and $|\phi_2(z)| \to 1$. Let $(z_n)_n$ be a sequence with $|\phi_1(z_n)| \to 1$ and $|\phi_2(z_n)| \to 1$ such that

$$\begin{aligned}
&\lim_n w(z_n)|\psi_1(z_n) - \psi_2(z_n)| \max\left\{\frac{1}{\tilde{v}(\phi_1(z_n))}, \frac{1}{\tilde{v}(\phi_2(z_n))}\right\} \\
&= \limsup_{\min\{|\phi_1(z)|, |\phi_2(z)|\} \to 1} w(z)|\psi_1(z) - \psi_2(z)| \max\left\{\frac{1}{\tilde{v}(\phi_1(z))}, \frac{1}{\tilde{v}(\phi_2(z))}\right\} .
\end{aligned}$$

If $\max\left\{\frac{1}{\tilde{v}(\phi_1(z_n))}, \frac{1}{\tilde{v}(\phi_2(z_n))}\right\} = \frac{1}{\tilde{v}(\phi_1(z_n))}$ we choose $(f_n)_n$ and $(g_n)_n$ as in the first case in the proof of (i) and set $h_n(z) := g_n(z)f_n(z)$. Then $h_n \in B_v^\infty$ and $h_n \to 0$ weakly in H_v^∞. Take a compact operator $K : H_v^\infty \to H_w^\infty$. Hence $\lim_n \|Kh_n\|_w = 0$. Thus we obtain

$$\begin{aligned}\|\psi_1 C_{\phi_1} - \psi_2 C_{\phi_2} - K\| &\geq \limsup_n w(z_n)\big|\psi_1(z_n)h_n(\phi_1(z_n)) - \psi_2(z_n)h_n(\phi_2(z_n))\big| \\ &\geq \limsup_n w(z_n)\big|\psi_1(z_n) - \psi_2(z_n)\big|\big|h_n(\phi_1(z_n))\big| \\ &\quad - \limsup_n w(z_n)\big|\psi_2(z_n)\big|\big|h_n(\phi_1(z_n)) - h_n(\phi_2(z_n))\big|\,.\end{aligned}$$

Now, if $\min\{|\psi_1(z), |\psi_2(z)|\} = |\psi_1(z_n)|$ take into account that by assumption $\frac{|\psi_1(z_n)|}{|\psi_2(z_n)|} \geq \alpha$. Using Lemma 1 and the boundedness of C_{ϕ_1,ψ_1} this yields

$$\begin{aligned}&\limsup_n w(z_n)\big|\psi_2(z_n)\big|\,\big|h_n(\phi_1(z_n)) - h_n(\phi_2(z_n))\big| \\ &\leq \frac{1}{\alpha}\limsup_n w(z_n)\big|\psi_1(z_n)\big|\,\rho(\phi_1(z_n), \phi_2(z_n))\frac{1}{\tilde{v}(\phi_1(z_n))} = 0\,.\end{aligned}$$

If $\min\{|\psi_1(z), |\psi_2(z)|\} = |\psi_2(z_n)|$, then obviously

$$\limsup_n w(z_n)|\psi_2(z_n)||h_n(\phi_1(z_n)) - h_n(\phi_2(z_n))| = 0\,.$$

Next we assume $\max\left\{\frac{1}{\tilde{v}(\phi_1(z_n))}, \frac{1}{\tilde{v}(\phi_2(z_n))}\right\} = \frac{1}{\tilde{v}(\phi_1(z_n))}$ and show the claim completely analogously to the previous case.

We now prove that there is a constant $C^* > 0$ such that $\max\{(i), (ii), (iii)\} \leq C^*\|C_{\phi_1,\psi_1} - C_{\phi_2,\psi_2}\|_e$. Take the sequence of linear operators $C_k : H(D) \to H(D)$, $k \in \mathbb{N}$, defined by $C_k f(z) = f(\frac{k}{k+1}z)$, which are continuous for the compact open topology and $C_k f \to f$ uniformly on every compact subset of D and the operators $C_k : H_v^\infty \to H_v^\infty$ are well-defined and compact with $||C_k|| \leq 1$.

For fixed $k \in \mathbb{N}$ we have,

$$||\psi_1 C_{\phi_1} - \psi_2 C_{\phi_2}||_e \leq ||(\psi_1 C_{\phi_1} - \psi_2 C_{\phi_2})(Id - C_k)||.$$

Let $f \in B_v^\infty$ and fix an arbitrary $r \in (0,1)$. Set $g_k := (Id - C_k)f$, so $g_k \in H_v^\infty$ and $||g_k||_v \leq 2$. Then

$$\begin{aligned}\big\|\psi_1 C_{\phi_1} - \psi_2 C_{\phi_2}\big\|_e &\leq \sup_{||f||_v \leq 1}\big\|(\psi_1 C_{\phi_1} - \psi_2 C_{\phi_2})g_k\big\|_w \\ &\leq \sup_{||f||_v \leq 1}\sup_{\{z;|\phi_1(z)|>r\}} w(z)\big|\psi_1(z)g_k(\phi_1(z)) - \psi_2(z)g_k(\phi_2(z))\big| \\ &\quad + \sup_{||f||_v \leq 1}\sup_{\{z;|\phi_2(z)|>r\}} w(z)\big|\psi_1(z)g_k(\phi_1(z)) - \psi_2(z)g_k(\phi_2(z))\big| \\ &\quad + \sup_{||f||_v \leq 1}\sup_{\{z;|\phi_1(z)|\leq r, |\phi_2(z)|\leq r\}} w(z)\big|\psi_1(z)g_k(\phi_1(z)) - \psi_2(z)g_k(\phi_2(z))\big| \\ &=: I_{k,r} + J_{k,r} + L_{k,r}.\end{aligned}$$

To estimate the first term $I_{k,r}$, for $z \in D$ with $|\phi_1(z)| > r$ we use Lemma 1 as in the proof of Theorem 2 to get,

$$\begin{aligned} w(z)|\psi_1(z)g_k(\phi_1(z)) - \psi_2(z)g_k(\phi_2(z))| \\ &\leq w(z)|\psi_1(z)|\,|g_k(\phi_1(z)) - g_k(\phi_2(z))| \\ &\quad + w(z)|\psi_1(z) - \psi_2(z)|\,|g_k(\phi_2(z))| \\ &\leq 2C_v|\psi_1(z)|\,w(z)\rho(\phi_1(z), \phi_2(z)) \max\left\{\frac{1}{\tilde{v}(\phi_1(z))}, \frac{1}{\tilde{v}(\phi_2(z))}\right\} \\ &\quad + w(z)|\psi_1(z) - \psi_2(z)|\,|g_k(\phi_2(z))|. \end{aligned}$$

Analogously we can estimate the term $J_{k,r}$.

The sequence of operators $(Id - C_k)_k$ satisfies $\lim_k (Id - C_k)g = 0$ for each g in $H(D)$, and the space $H(D)$ endowed with the compact open topology co is a Frećhet space. By the Banach-Steinhaus theorem, $(Id - C_k)_k$ converges to zero uniformly on the compact subsets of $(H(D), co)$. Since the closed unit ball of H_v^∞ is a compact subset of $(H(D), co)$ we conclude that

$$\lim_k \sup_{||f||_v \leq 1} \sup_{|\xi| \leq r} |((Id - C_k)f)(\xi)| = 0.$$

If $|\phi_2(z)| \leq r$ in the term $I_{k,r}$, then we conclude that

$$\lim_{r\to 1} \limsup_k I_{k,r} \leq 2 \limsup_{|\phi_1(z)|\to 1} w(z)\, \frac{|\psi_1(z)|}{\tilde{v}(\phi_1(z))}\, \rho(\phi_1(z), \phi_2(z)).$$

In the case $|\phi_2(z)| > r$, we have that

$$\begin{aligned} \lim_{r\to 1} \limsup_k I_{k,r} \leq 2 &\limsup_{\min\{|\phi_1(z)|, |\phi_2(z)|\}\to 1} w(z)|\psi_1(z) - \psi_2(z)| \max\left\{\frac{1}{\tilde{v}(\phi_1(z))}, \frac{1}{\tilde{v}(\phi_2(z))}\right\} \\ &+ 2 \limsup_{|\phi_1(z)|\to 1} w(z)|\psi_1(z)|\, \rho(\phi_1(z), \phi_2(z)) \max\left\{\frac{1}{\tilde{v}(\phi_1(z))}, \frac{1}{\tilde{v}(\phi_2(z))}\right\}. \end{aligned}$$

Analogously we consider the cases $|\phi_1(z)| \leq r$ and $|\phi_1(z)| > r$ in the term $J_{k,r}$.

Since $\psi_1, \psi_2 \in H_w^\infty$, we have that $\lim_{r\to 1} \limsup_k L_{k,r} = 0$, and the statement follows. $\square$

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Summary

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