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An algorithm for lifting points in a tropical variety

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Abstract

The aim of this paper is to give a constructive proof of one of the basic theorems of tropical geometry: given a point on a tropical variety (defined using initial ideals), there exists a Puiseux-valued "lift" of this point in the algebraic variety. This theorem is so fundamental because it justifies why a tropical variety (defined combinatorially using initial ideals) carries information about algebraic varieties: it is the image of an algebraic variety over the Puiseux series under the valuation map. We have implemented the "lifting algorithm" using Singular and `Gfan` if the base field is $\mathbb{Q}$. As a byproduct we get an algorithm to compute the Puiseux expansion of a space curve singularity in $(K^{n+1}, 0)$.

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1. Introduction

In tropical geometry, algebraic varieties are replaced by certain piecewise linear objects called tropical varieties. Many algebraic geometry theorems have been "translated" to the tropical world (see for example [18, 25, 22, 8] and many more). Because new methods can be used in the tropical world — for example, combinatorial methods — and because the objects seem easier to deal with due to their piecewise linearity, tropical geometry is a promising tool for deriving new results in algebraic geometry. (For example, the Welschinger invariant can be computed tropically, see [18]).

There are two ways to define the tropical variety $\mathrm{Trop}(J)$ for an ideal J in the polynomial ring $K\{\{t\}\}[x_1, \ldots, x_n]$ over the field of Puiseux series (see Definition 2.1). One way is to define the tropical variety combinatorially using t-initial ideals (see Definition 2.4 and Definition 2.10, resp. [22]) — this definition is more helpful when computing and it is the definition we use in this paper. The other way to define tropical varieties is as the closure of the image of the algebraic variety $V(J)$ of J in $K\{\{t\}\}^n$ under the negative of the valuation map (see Remark 2.2, resp. [21, Definition 2.1]) — this gives more insight why tropical varieties carry information about algebraic varieties.

It is our main aim in this paper to give a constructive proof that these two concepts coincide (see Theorem 2.13), and to derive that way an algorithm which allows to lift a given point $\omega \in \mathrm{Trop}(J)$ to a point in $V(J)$ up to given order (see Algorithms 3.8 and 4.8). The algorithm has been implemented using the commutative algebra system SINGULAR (see [10]) and the programme `Gfan` (see [11]), which computes Gröbner fans and tropical varieties.

Theorem 2.13 has been proved in the case of a principal ideal by [6, Theorem 2.1.1]. There is also a constructive proof for a principal ideal in [24, Theorem 2.4]. For the general case, there is a proof in [23, Theorem 2.1], which has a gap however. Furthermore, there is a proof in [5, Theorem 4.2], using affinoid algebras, and in [12, Lemma 5.2.2], using flat schemes. A more general statement is proved in [20, Theorem 4.2]. Our proof has the advantage that it is constructive and works for an arbitrary ideal J.

We describe our algorithm first in the case where the ideal is 0-dimensional. This algorithm can be viewed as a variant of an algorithm presented by Joseph Maurer in [17], a paper from 1980. In fact, he uses the term "critical tropism" for a point in the tropical variety, even though tropical varieties were not defined by that time. Apparently, the notion goes back to Monique Lejeune-Jalabert and Bernard Teissier[1] (see [14]).

[1] Asked about this coincidence in the two notions Bernard Teissier sent us the following kind and interesting explanation: *As far as I know the term did not exist before. We tried to convey the idea that giving different weights to some variables made the space "anisotropic", and we were intrigued by the structure, for example, of anisotropic projective spaces (which are nowadays called weighted projective spaces). From there to "tropismes critiques" was a quite natural linguistic movement. Of course there was no "tropical" idea around, but as you say, it is an amusing coincidence. The Greek "Tropos" usually designates change, so that "tropisme critique" is well adapted to denote the values where the change of weights becomes critical for the computation of the initial ideal. The term "Isotropic", apparently due to Cauchy, refers to the property of presenting the same (physical) characters in all directions. Anisotropic is, of course, its negation. The name of Tropical geometry originates, as you probably know, from tropical algebra which honours the Brazilian computer scientist Imre Simon living close to the tropics, where the course of the sun changes back to the equator. In a way the tropics of Capricorn and Cancer represent, for the sun, critical tropisms.*

This paper is organised as follows: In Section 2 we recall basic definitions and state the main result. In Section 3 we give a constructive proof of the main result in the 0-dimensional case and deduce an algorithm. In Section 4 we reduce the arbitrary case algorithmically to the 0-dimensional case, and in Section 5 we gather some simple results from commutative algebra for the lack of a better reference. The proofs of both cases heavily rely on a good understanding of the relation of the dimension of an ideal J over the Puiseux series with its t-initial ideal, respectively with its restriction to the rings $R_N[\underline{x}]$ introduced below (see Definition 2.1). This will be studied in Section 6. Some of the theoretical as well as the computational results use Theorem 2.8 which was proved in [15] using standard bases in the mixed power series polynomial ring $K[[t]][\underline{x}]$. We give an alternative proof in Section 7.

We would like to thank Bernd Sturmfels for suggesting the project and for many helpful discussions, and Michael Brickenstein, Gerhard Pfister and Hans Schönemann for answering many questions concerning SINGULAR. Also we would like to thank Sam Payne for helpful remarks and for pointing out a mistake in an earlier version of this paper.

Our programme can be downloaded from the web page

www.mathematik.uni-kl.de/~keilen/en/tropical.html.

2. Basic notations and the main theorem

In this section we will introduce the basic notations used throughout the paper.

DEFINITION 2.1 Let K be an arbitrary field. We consider for $N \in \mathbb{N}_{>0}$ the discrete valuation ring

$$R_N = K[[t^{1/N}]] = \left\{ \sum_{\alpha=0}^{\infty} a_\alpha \cdot t^{\alpha/N} \;\middle|\; a_\alpha \in K \right\}$$

of formal power series in the unknown $t^{1/N}$ with *discrete valuation*

$$\operatorname{val}\left(\sum_{\alpha=0}^{\infty} a_\alpha \cdot t^{\alpha/N}\right) = \operatorname{ord}_t\left(\sum_{\alpha=0}^{\infty} a_\alpha \cdot t^{\alpha/N}\right) = \min\left\{\frac{\alpha}{N} \;\middle|\; a_\alpha \neq 0\right\} \in \frac{1}{N}\cdot\mathbb{Z}\cup\{\infty\},$$

and we denote by $L_N = \operatorname{Quot}(R_N)$ its quotient field. If $N \mid M$ then in an obvious way we can think of R_N as a subring of R_M, and thus of L_N as a subfield of L_M. We call the direct limit of the corresponding direct system

$$L = K\{\{t\}\} = \varinjlim L_N = \bigcup_{N>0} L_N$$

the *field of (formal) Puiseux series* over K.

Recall that if K is algebraically closed of characteristic 0, then L is algebraically closed.

Remark 2.2 If $0 \neq N \in \mathbb{N}$ then $S_N = \{1, t^{1/N}, t^{2/N}, t^{3/N}, \dots\}$ is a multiplicatively closed subset of R_N, and obviously

$$L_N = S_N^{-1} R_N = \left\{ t^{-\alpha/N} \cdot f \;\middle|\; f \in R_N, \alpha \in \mathbb{N} \right\}.$$

The valuation of R_N extends to L_N, and thus L, by $\mathrm{val}\left(\frac{f}{g}\right) = \mathrm{val}(f) - \mathrm{val}(g)$ for $f, g \in R_N$ with $g \neq 0$. In particular, $\mathrm{val}(0) = \infty$.

Notation 2.3 Since an ideal $J \trianglelefteq L[\underline{x}]$ is generated by finitely many elements, the set

$$\mathcal{N}(J) = \{N \in \mathbb{N}_{>0} \mid \langle J \cap R_N[\underline{x}] \rangle_{L[\underline{x}]} = J\}$$

is non-empty, and if $N \in \mathcal{N}(J)$ then $N \cdot \mathbb{N}_{>0} \subseteq \mathcal{N}(J)$. We also introduce the notation $J_{R_N} = J \cap R_N[\underline{x}]$.

Remark and Definition 2.4 Let $N \in \mathbb{N}_{>0}$, $w = (w_0, \dots, w_n) \in \mathbb{R}_{<0} \times \mathbb{R}^n$, and $q \in \mathbb{R}$.

We may consider the direct product

$$V_{q,w,N} = \prod_{\substack{(\alpha,\beta) \in \mathbb{N}^{n+1} \\ w \cdot (\frac{\alpha}{N}, \beta) = q}} K \cdot t^{\alpha/N} \cdot \underline{x}^\beta$$

of K-vector spaces and its subspace

$$W_{q,w,N} = \bigoplus_{\substack{(\alpha,\beta) \in \mathbb{N}^{n+1} \\ w \cdot (\frac{\alpha}{N}, \beta) = q}} K \cdot t^{\alpha/N} \cdot \underline{x}^\beta.$$

As a K-vector space the formal power series ring $K[[t^{1/N}, \underline{x}]]$ is just

$$K[[t^{1/N}, \underline{x}]] = \prod_{q \in \mathbb{R}} V_{q,w,N},$$

and we can thus write any power series $f \in K[[t^{1/N}, \underline{x}]]$ in a unique way as

$$f = \sum_{q \in \mathbb{R}} f_{q,w} \quad \text{with} \quad f_{q,w} \in V_{q,w,N}.$$

Note that this representation is independent of N in the sense that if $f \in K[[t^{1/N'}, \underline{x}]]$ for some other $N' \in \mathbb{N}_{>0}$ then we get the same non-vanishing $f_{q,w}$ if we decompose f with respect to N'.

Moreover, if $0 \neq f \in R_N[\underline{x}] \subset K[[t^{1/N}, \underline{x}]]$, then there is a *maximal* $\hat{q} \in \mathbb{R}$ such that $f_{\hat{q},w} \neq 0$ and $f_{q,w} \in W_{q,w,N}$ for all $q \in \mathbb{R}$, since the $\underline{x}$-degree of the monomials involved in f is bounded. We call the elements $f_{q,w}$ *w-quasihomogeneous* of w-degree $\deg_w(f_{q,w}) = q \in \mathbb{R}$,

$$\mathrm{in}_w(f) := f_{\hat{q},w} \in K[t^{1/N}, \underline{x}]$$

the *w-initial form* of f, and

$$\mathrm{ord}_w(f) := \hat{q} = \max\{\deg_w(f_{q,w}) \mid f_{q,w} \neq 0\}$$

the *w-order* of f. Set $\in_\omega(0) = 0$. If $t^\beta x^\alpha \neq t^{\beta'} x^{\alpha'}$ are both monomials of $\mathrm{in}_w(f)$, then $\alpha \neq \alpha'$.

For $I \subseteq R_N[\underline{x}]$ we call

$$\mathrm{in}_w(I) = \langle \mathrm{in}_w(f) \mid f \in I \rangle \trianglelefteq K[t^{1/N}, \underline{x}]$$

the *w-initial ideal* of I. Note that its definition depends on N.

Moreover, we call for $f \in R_N[\underline{x}]$

$$\mathrm{t} - \mathrm{in}_w(f) = \mathrm{in}_w(f)(1, \underline{x}) = \mathrm{in}_w(f)_{|t=1} \in K[\underline{x}]$$

the *t-initial form of f w.r.t. w*, and if $f = t^{-\alpha/N} \cdot g \in L[\underline{x}]$ with $g \in R_N[\underline{x}]$ we set

$$\mathrm{t} - \mathrm{in}_w(f) := \mathrm{t} - \mathrm{in}_w(g).$$

This definition does not depend on the particular representation of f.

If $J \subseteq L[\underline{x}]$ is a subset of $L[\underline{x}]$, then

$$\mathrm{t} - \mathrm{in}_w(J) = \langle \mathrm{t} - \mathrm{in}_w(f) \mid f \in J \rangle \trianglelefteq K[\underline{x}]$$

is the *t-initial ideal* of J, which does not depend on any N.

For two w-quasihomogeneous elements $f_{q,w} \in W_{q,w,N}$ and $f_{q',w} \in W_{q',w,N}$ we have $f_{q,w} \cdot f_{q',w} \in W_{q+q',w,N}$. In particular, $\mathrm{in}_w(f \cdot g) = \mathrm{in}_w(f) \cdot \mathrm{in}_w(g)$ for $f, g \in R_N[\underline{x}]$, and $\mathrm{t} - \mathrm{in}_w(f \cdot g) = \mathrm{t} - \mathrm{in}_w(f) \cdot \mathrm{t} - \mathrm{in}_w(g)$ for $f, g \in L[\underline{x}]$.

Example 2.5 Let $w = (-1, -2, -1)$ and

$$f = (2t + t^{3/2} + t^2) \cdot x^2 + (-3t^3 + 2t^4) \cdot y^2 + t^5 xy^2 + (t + 3t^2) \cdot x^7 y^2.$$

Then $\mathrm{ord}_w(f) = -5$, $\mathrm{in}_w(f) = 2tx^2 - 3t^3y^2$, and $\mathrm{t} - \mathrm{in}_w(f) = 2x^2 - 3y^2$.

Notation 2.6 Throughout this paper we will mostly use the weight -1 for the variable t, and in order to simplify the notation we will then usually write for $\omega \in \mathbb{R}^n$

$$\mathrm{in}_\omega \quad \text{instead of} \quad \mathrm{in}_{(-1,\omega)}$$

and

$$\mathrm{t} - \mathrm{in}_\omega \quad \text{instead of} \quad \mathrm{t} - \mathrm{in}_{(-1,\omega)}.$$

The case that $\omega = (0, \ldots, 0)$ is of particular interest, and we will simply write

$$\mathrm{in}_0 \quad \text{respectively} \quad \mathrm{t} - \mathrm{in}_0.$$

This should not lead to any ambiguity.

In general, the t-initial ideal of an ideal J is not generated by the t-initial forms of the given generators of J.

Example 2.7 Let $J = \langle tx + y, x + t \rangle \lhd L[x, y]$ and $\omega = (1, -1)$. Then $y - t^2 \in J$, but

$$y = \mathrm{t} - \mathrm{in}_\omega(y - t^2) \notin \langle \mathrm{t} - \mathrm{in}_\omega(tx + y), \mathrm{t} - \mathrm{in}_\omega(x + t) \rangle = \langle x \rangle.$$

We can compute the t-initial ideal using standard bases by [15, Corollary 6.11].

Theorem 2.8

Let $J = \langle I \rangle_{L[\underline{x}]}$ with $I \trianglelefteq K[t^{1/N}, \underline{x}]$, $\omega \in \mathbb{Q}^n$ and G be a standard basis of I with respect to $>_\omega$ (see Remark 3.7 for the definition of $>_\omega$).

Then $\mathrm{t}-\mathrm{in}_\omega(J) = \mathrm{t}-\mathrm{in}_\omega(I) = \langle \mathrm{t}-\mathrm{in}_\omega(G) \rangle \trianglelefteq K[\underline{x}]$.

The proof of this theorem uses standard basis techniques in the ring $K[[t]][\underline{x}]$. We give an alternative proof in Section 7.

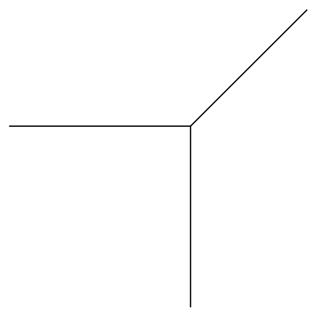
EXAMPLE 2.9 In Example 2.7, $G = (tx + y, x + t, y - t^2)$ is a suitable standard basis and thus $\mathrm{t}-\mathrm{in}_\omega(J) = \langle x, y \rangle$.

DEFINITION 2.10 Let $J \trianglelefteq L[\underline{x}]$ be an ideal then the *tropical variety* of J is defined as

$$\mathrm{Trop}(J) = \{\omega \in \mathbb{R}^n \mid \mathrm{t}-\mathrm{in}_\omega(J) \text{ is monomial free}\}.$$

It is possible that $\mathrm{Trop}(J) = \emptyset$.

EXAMPLE 2.11 Let $J = \langle x + y + 1 \rangle \subset L[x, y]$. As J is generated by one polynomial f which then automatically is a standard basis, the t-initial ideal $\mathrm{t}-\mathrm{in}_\omega(J)$ will be generated by $\mathrm{t}-\mathrm{in}_\omega(f)$ for any ω. Hence $\mathrm{t}-\mathrm{in}_\omega(J)$ contains no monomial if and only if $\mathrm{t}-\mathrm{in}_\omega(f)$ is not a monomial. This is the case for all ω such that $\omega_1 = \omega_2 \geq 0$, or $\omega_1 = 0 \geq \omega_2$, or $\omega_2 = 0 \geq \omega_1$. Hence the tropical variety $\mathrm{Trop}(J)$ looks as follows:



We need the following basic results about tropical varieties.

Lemma 2.12

Let $J, J_1, \ldots, J_k \trianglelefteq L[\underline{x}]$ be ideals. Then:

(a) $J_1 \subseteq J_2 \implies \mathrm{Trop}(J_1) \supseteq \mathrm{Trop}(J_2)$,
(b) $\mathrm{Trop}(J_1 \cap \ldots \cap J_k) = \mathrm{Trop}(J_1) \cup \ldots \cup \mathrm{Trop}(J_k)$,
(c) $\mathrm{Trop}(J) = \mathrm{Trop}(\sqrt{J}) = \bigcup_{P \in \mathrm{minAss}(J)} \mathrm{Trop}(P)$, *and*
(d) $\mathrm{Trop}(J_1 + J_2) \subseteq \mathrm{Trop}(J_1) \cap \mathrm{Trop}(J_2)$.

Proof. Suppose that $J_1 \subseteq J_2$ and $\omega \in \mathrm{Trop}(J_2) \setminus \mathrm{Trop}(J_1)$. Then $\mathrm{t}-\mathrm{in}_\omega(J_1)$ contains a monomial, but since $\mathrm{t}-\mathrm{in}_\omega(J_1) \subseteq \mathrm{t}-\mathrm{in}_\omega(J_2)$ this contradicts $\omega \in \mathrm{Trop}(J_2)$. Thus $\mathrm{Trop}(J_2) \subseteq \mathrm{Trop}(J_1)$. This shows (a).

Since $J_1 \cap \ldots \cap J_k \subseteq J_i$ for each $i = 1, \ldots, k$ the first assertion implies that

$$\mathrm{Trop}(J_1) \cup \ldots \cup \mathrm{Trop}(J_k) \subseteq \mathrm{Trop}(J_1 \cap \ldots \cap J_k).$$

Conversely, if $\omega \notin \mathrm{Trop}(J_i)$ for $i = 1, \ldots, k$ then there exist polynomials $f_i \in J_i$ such that $\mathrm{t}-\mathrm{in}_\omega(f_i)$ is a monomial. But then $\mathrm{t}-\mathrm{in}_\omega(f_1 \cdots f_k) = \mathrm{t}-\mathrm{in}_\omega(f_1) \cdots \mathrm{t}-\mathrm{in}_\omega(f_k)$ is a monomial and $f_1 \cdots f_k \in J_1 \cdots J_k \subseteq J_1 \cap \ldots \cap J_k$. Thus $\omega \notin \mathrm{Trop}(J_1 \cap \ldots \cap J_k)$, which shows (b).

For (c) it suffices to show that $\mathrm{Trop}(J) \subseteq \mathrm{Trop}(\sqrt{J})$, since $J \subseteq \sqrt{J} = \bigcap_{P \in \mathrm{minAss}(J)} P$. If $\omega \notin \mathrm{Trop}(\sqrt{J})$ then there is an $f \in \sqrt{J}$ such that $\mathrm{t}-\mathrm{in}_\omega(f)$ is a monomial and such that $f^m \in J$ for some m. But then $\mathrm{t}-\mathrm{in}_\omega(f^m) = \mathrm{t}-\mathrm{in}_\omega(f)^m$ is a monomial and thus $\omega \notin \mathrm{Trop}(J)$.

Finally (d) is obvious from the definition. $\square$

We are now able to state our main theorem.

Theorem 2.13

If K is algebraically closed of characteristic zero and $J \trianglelefteq K\{\{t\}\}[\underline{x}]$ is an ideal then

$$\omega \in \mathrm{Trop}(J) \cap \mathbb{Q}^n \qquad \Longleftrightarrow \qquad \exists\, p \in V(J):\ -\mathrm{val}(p) = \omega \in \mathbb{Q}^n,$$

where val *is the coordinate-wise valuation.*

The proof of one direction is straight forward and it does not require that K is algebraically closed.

Proposition 2.14

If $J \trianglelefteq L[\underline{x}]$ is an ideal and $p \in V(J) \cap (L^*)^n$, then $-\mathrm{val}(p) \in \mathrm{Trop}(J)$.

Proof. Let $p = (p_1, \ldots, p_n)$, and let $\omega = -\mathrm{val}(p) \in \mathbb{Q}^n$. If $f \in J$, we have to show that $\mathrm{t}-\mathrm{in}_\omega(f)$ is not a monomial, but since this property is preserved when multiplying with some $t^{\frac{\alpha}{N}}$ we may as well assume that $f \in J_{R_N}$. As $p \in V(J)$, we know that $f(p) = 0$. In particular the terms of lowest t-order in $f(p)$ have to cancel. But the terms of lowest order in $f(p)$ are $\mathrm{in}_\omega(f)(a_1 \cdot t^{-\omega_1}, \ldots, a_n \cdot t^{-\omega_n})$, where $p_i = a_i \cdot t^{-\omega_i} + h.o.t.$. Hence $\mathrm{in}_\omega(f)(a_1 t^{-\omega_1}, \ldots, a_n t^{-\omega_n}) = 0$, which is only possible if $\mathrm{in}_\omega(f)$, and thus $\mathrm{t}-\mathrm{in}_\omega(f)$, is not a monomial. $\square$

Essentially, this was shown by Newton in [19].

Remark 2.15 If the base field K in Theorem 2.13 is not algebraically closed or not of characteristic zero, then the Puiseux series field is not algebraically closed (see e.g. [13]). We therefore cannot expect to be able to lift each point in the tropical variety of an ideal $J \triangleleft K\{\{t\}\}[\underline{x}]$ to a point in $V(J) \subseteq K\{\{t\}\}^n$. However, if we replace $V(J)$ by the vanishing set, say W, of J over the algebraic closure $\overline{L}$ of $K\{\{t\}\}$ then it is still true that each point ω in the tropical variety of J can be lifted to a point $p \in W$ such that $\mathrm{val}(p) = -\omega$. For this we note first that if $\dim(J) = 0$ then the non-constructive proof of Theorem 3.1 works by passing from J to $\langle J \rangle_{\overline{L}[\underline{x}]}$, taking into account that the non-archimedean valuation of a field in a natural way extends to its algebraic closure. And if $\dim(J) > 0$ then we can add generators to J by Proposition 4.6 and

Remark 4.5 so as to reduce to the zero dimensional case before passing to the algebraic closure of $K\{\{t\}\}$.

Note, it is even possible to apply Algorithm 3.8 in the case of positive characteristic. However, due to the weird nature of the algebraic closure of the Puiseux series field in that case we cannot guarantee that the result will coincide with a solution of J up to the order up to which it is computed. It may very well be the case that some intermediate terms are missing (see [13, Section 5]).

3. Zero-dimensional lifting lemma

In this section we want to give a constructive proof of the lifting Lemma 3.1.

Theorem 3.1 (lifting Lemma)

Let K be an algebraically closed field of characteristic zero and $L = K\{\{t\}\}$. If $J \lhd L[\underline{x}]$ is a zero dimensional ideal and $\omega \in \mathrm{Trop}(J) \cap \mathbb{Q}^n$, then there is a point $p \in V(J)$ such that $-\mathrm{val}(p) = \omega$.

Non-Constructive Proof. If $\omega \in \mathrm{Trop}(J)$ then by Lemma 2.12 there is an associated prime $P \in \mathrm{minAss}(J)$ such that $\omega \in \mathrm{Trop}(P)$. But since $\dim(J) = 0$ the ideal P is necessarily a maximal ideal, and since L is algebraically closed it is of the form

$$P = \langle x_1 - p_1, \ldots, x_n - p_n \rangle$$

with $p_1, \ldots, p_n \in L$. Since $\omega \in \mathrm{Trop}(P)$ the ideal $\mathrm{t} - \mathrm{in}_\omega(P)$ does not contain any monomial, and therefore necessarily $\mathrm{ord}_t(p_i) = -\omega_i$ for all $i = 1, \ldots, n$. This shows that $p = (p_1, \ldots, p_n) \in V(P) \subseteq V(J)$ and $\mathrm{val}(p) = -\omega$. □

The drawback of this proof is that in order to find p one would have to be able to find the associated primes of J which would amount to something close to primary decomposition over L. This is of course not feasible. We will instead adapt the constructive proof that L is algebraically closed, i.e. the Newton-Puiseux Algorithm for plane curves, which has already been generalised to space curves (see [17, 1]) to our situation in order to compute the point p up to any given order.

The idea behind this is very simple and the first recursion step was basically already explained in the proof of Proposition 2.14. Suppose we have a polynomial $f \in R_N[\underline{x}]$ and a point

$$p = \Big(u_1 \cdot t^{\alpha_1} + v_1 \cdot t^{\beta_1} + \ldots, \ldots, u_n \cdot t^{\alpha_n} + v_n \cdot t^{\beta_n} + \ldots\Big) \in (L^*)^n.$$

Then, a priori, the term of lowest t-order in $f(p)$ will be $\mathrm{in}_{-\alpha}(f)(u_1 \cdot t^{\alpha_1}, \ldots, u_n \cdot t^{\alpha_n})$. Thus, in order for $f(p)$ to be zero it is necessary that $\mathrm{t} - \mathrm{in}_{-\alpha}(f)(u_1, \ldots, u_n) = 0$. Let p' denote the tail of p, that is $p_i = u_i \cdot t^{\alpha_i} + t^{\alpha_i} \cdot p_i'$. Then p' is a zero of

$$f' = f\big(t^{\alpha_1} \cdot (u_1 + x_1), \ldots, t^{\alpha_n} \cdot (u_n + x_n)\big).$$

The same arguments then show that $\mathrm{t} - \mathrm{in}_{\alpha-\beta}(f')(v_1, \ldots, v_n) = 0$, and assuming now that none of the v_i is zero we find $\mathrm{t} - \mathrm{in}_{\alpha-\beta}(f')$ must be monomial free, that is $\alpha - \beta$ is a point in the tropical variety and all its components are strictly negative.

The basic idea for the algorithm which computes a suitable p is thus straight forward. Given $\omega = -\alpha$ in the tropical variety of an ideal J, compute a point $u \in V(\mathrm{t}-\mathrm{in}_\omega(J))$ apply the above transformation to J and compute a negative-valued point in the tropical variety of the transformed ideal. Then go on recursively.

It may happen that the solution that we are about to construct this way has some component with only finitely many terms. Then after a finite number of steps there might be no suitable ω in the tropical variety. However, in that situation we can simply eliminate the corresponding variable for the further computations.

Example 3.2 Consider the ideal $J = \langle f_1, \ldots, f_4 \rangle \lhd L[x, y]$ with

$$\begin{aligned} f_1 =&\ y^2 + 4t^2 y + (-t^3 + 2t^4 - t^5), \\ f_2 =&\ (1+t) \cdot x - y + (-t - 3t^2), \\ f_3 =&\ xy + (-t + t^2) \cdot x + (t^2 - t^4), \\ f_4 =&\ x^2 - 2tx + (t^2 - t^3). \end{aligned}$$

The t-initial ideal of J with respect to $\omega = \big(-1, -\frac{3}{2}\big)$ is

$$\mathrm{t} - \mathrm{in}_\omega(J) = \langle y^2 - 1, x - 1 \rangle,$$

so that $\omega \in \mathrm{Trop}(J)$ and $u = (1, 1)$ is a suitable choice. Applying the transformation $\gamma_{\omega,u} : (x, y) \mapsto \big(t \cdot (1 + x), t^{3/2} \cdot (1 + y)\big)$ to J we get $J' = \langle f_1', \ldots, f_4' \rangle$ with

$$\begin{aligned} f_1' &= t^3 y^2 + \big(2t^3 + 4t^{7/2}\big) \cdot y + \big(4t^{7/2} + 2t^4 - t^5\big), \\ f_2' &= (t + t^2) \cdot x - t^{3/2} \cdot y + \big(-t^{3/2} - 2t^2\big), \\ f_3' &= t^{5/2} \cdot xy + \big(-t^2 + t^3 + t^{5/2}\big) \cdot x + t^{5/2} \cdot y + \big(t^{5/2} + t^3 - t^4\big), \\ f_4' &= t^2 x^2 - t^3. \end{aligned}$$

This shows that the x-coordinate of a solution of J' necessarily is $x = \pm t^{1/2}$, and we could substitute this for x in the other equations in order to reduce by one variable. We will instead see what happens when we go on with our algorithm.

The t-initial ideal of J' with respect to $\omega' = \big(-\frac{1}{2}, -\frac{1}{2}\big)$ is

$$\mathrm{t} - \mathrm{in}_{\omega'}(J') = \langle y + 2, x - 1 \rangle,$$

so that $\omega' \in \mathrm{Trop}(J')$ and $u' = (1, -2)$ is our only choice. Applying the transformation $\gamma_{\omega',u'} : (x, y) \mapsto \big(t^{1/2} \cdot (1 + x), t^{1/2} \cdot (-2 + y)\big)$ to J' we get the ideal $J'' = \langle f_1'', \ldots, f_4'' \rangle$ with

$$\begin{aligned} f_1'' &= t^4 y^2 + 2t^{7/2} y + \big(-2t^4 - t^5\big), \\ f_2'' &= \big(t^{3/2} + t^{5/2}\big) \cdot x - t^2 \cdot y + t^{5/2}, \\ f_3'' &= t^{7/2} \cdot xy + \big(-t^{5/2} + t^3 - t^{7/2}\big) \cdot x + \big(t^3 + t^{7/2}\big) \cdot y + \big(-t^{7/2} - t^4\big), \\ f_4'' &= t^3 x^2 + 2t^3 x. \end{aligned}$$

If we are to find an $\omega'' \in \mathrm{Trop}(J'')$, then f_4'' implies that necessarily $\omega_1'' = 0$. But we are looking for an ω'' all of whose entries are strictly negative. The reason why this does not exist is that there is a solution of J'' with $x = 0$. We thus have to eliminate the variable x, and replace J'' by the ideal $J''' = \langle f''' \rangle$ with

$$f''' = y - t^{1/2}.$$

Then $\omega''' = -\frac{1}{2} \in \mathrm{Trop}(J''')$ and $\mathrm{t-in}_{\omega'''}(f''') = y-1$. Thus $u''' = 1$ is our only choice, and since $f'''(u''' \cdot t^{-\omega'''}) = f'''(t^{1/2}) = 0$ we are done.

Backwards substitution gives

$$\begin{aligned} p &= \left(t^{\omega_1} \cdot \left(u_1 + t^{\omega_1'} \cdot (u_1' + 0)\right), t^{\omega_2} \cdot \left(u_2 + t^{\omega_2'} \cdot \left(u_2' + t^{\omega_2'''} \cdot u'''\right)\right)\right) \\ &= \left(t \cdot \left(1 + t^{1/2}\right), t^{3/2} \cdot \left(1 + t^{1/2} \cdot \left(-2 + t^{1/2}\right)\right)\right) \\ &= \left(t + t^{3/2}, t^{3/2} - 2t^2 + t^{5/2}\right) \end{aligned}$$

as a point in $V(J)$ with $\mathrm{val}(p) = (1, \frac{3}{2}) = -\omega$. Note that in general the procedure will not terminate.

For the proof that this algorithm works we need two types of transformations which we are now going to introduce and study.

Definition and Remark 3.3 For $\omega' \in \mathbb{Q}^n$ let us consider the L-algebra isomorphism

$$\Phi_{\omega'} : L[\underline{x}] \longrightarrow L[\underline{x}] : x_i \mapsto t^{-\omega_i'} \cdot x_i,$$

and the isomorphism which it induces on L^n

$$\phi_{\omega'} : L^n \to L^n : (p_1', \ldots, p_n') \mapsto (t^{-\omega_1'} \cdot p_1', \ldots, t^{-\omega_n'} \cdot p_n').$$

Suppose we have found a $p' \in V\big(\Phi_{\omega'}(J)\big)$, then $p = \phi_{\omega'}(p') \in V(J)$ and $\mathrm{val}(p) = \mathrm{val}(p') - \omega'$.

Thus choosing ω' appropriately we may in Theorem 3.1 assume that $\omega \in \mathbb{Q}^n_{<0}$, which due to Corollary 6.15 implies that the dimension of J behaves well when contracting to the power series ring $R_N[\underline{x}]$ for a suitable N.

Note also the following properties of $\Phi_{\omega'}$, which we will refer to quite frequently. If $J \trianglelefteq L[\underline{x}]$ is an ideal, then

$$\dim(J) = \dim\left(\Phi_{\omega'}(J)\right) \text{ and } \mathrm{t-in}_{\omega'}(J) = \mathrm{t-in}_0(\Phi_{\omega'}(J)),$$

where the latter is due to the fact that

$$\deg_w\left(t^\alpha \cdot \underline{x}^\beta\right) = -\alpha + \omega' \cdot \beta = \deg_v\left(t^{\alpha - \omega' \cdot \beta} \cdot \underline{x}^\beta\right) = \deg_v\left(\Phi_{\omega'}(t^\alpha \cdot \underline{x}^\beta)\right)$$

with $w = (-1, \omega')$ and $v = (-1, 0, \ldots, 0)$.

Definition and Remark 3.4 For $u = (u_1, \ldots, u_n) \in K^n$, $\omega \in \mathbb{Q}^n$ and $w = (-1, \omega)$ we consider the L-algebra isomorphism

$$\gamma_{\omega,u} : L[\underline{x}] \longrightarrow L[\underline{x}] : x_i \mapsto t^{-\omega_i} \cdot (u_i + x_i),$$

and its effect on a w-quasihomogeneous element

$$f_{q,w} = \sum_{\substack{(\alpha,\beta) \in \mathbb{N}^{n+1} \\ -\frac{\alpha}{N} + \omega \cdot \beta = q}} a_{\alpha,\beta} \cdot t^{\alpha/N} \cdot \underline{x}^\beta.$$

If we set

$$p_\beta := \prod_{i=1}^{n}(u_i + x_i)^{\beta_i} - u^\beta \in \langle x_1, \ldots, x_n\rangle \lhd K[\underline{x}]$$

then

$$\begin{aligned}\gamma_{\omega,u}(f_{q,w}) &= \sum_{-\frac{\alpha}{N}+\omega\cdot\beta=q} a_{\alpha,\beta}\cdot t^{\alpha/N}\cdot\prod_{i=1}^{n} t^{-\omega_i\cdot\beta_i}\cdot(u_i+x_i)^{\beta_i}\\ &= t^{-q}\cdot\sum_{-\frac{\alpha}{N}+\omega\cdot\beta=q} a_{\alpha,\beta}\cdot(u^\beta+p_\beta)\\ &= t^{-q}\cdot\Big(f_{q,w}(1,u)+\sum_{-\frac{\alpha}{N}+\omega\cdot\beta=q} a_{\alpha,\beta}\cdot p_\beta\Big)\\ &= t^{-q}\cdot f_{q,w}(1,u)+t^{-q}\cdot p_{f_{q,w},u},\end{aligned} \tag{1}$$

with

$$p_{f_{q,w},u} := \sum_{-\frac{\alpha}{N}+w\cdot\beta=q} a_{\alpha,\beta}\cdot p_\beta \in \langle x_1,\ldots,x_n\rangle \lhd K[\underline{x}].$$

In particular, if $\omega \in \frac{1}{N}\cdot\mathbb{Z}^n$ and $f = \sum_{q\le\hat{q}} f_{q,w} \in R_N[\underline{x}]$ with $\hat{q} = \mathrm{ord}_\omega(f)$ then

$$\gamma_{\omega,u}(f) = t^{-\hat{q}}\cdot g$$

where

$$g = \sum_{q\le\hat{q}}\left(t^{\hat{q}-q}\cdot f_{q,w}(1,u) + t^{\hat{q}-q}\cdot p_{f_{q,w},u}\right) \in R_N[\underline{x}].$$

□

The following lemma shows that if we consider the transformed ideal $\gamma_{\omega,u}(J)\cap R_N[\underline{x}]$ in the power series ring $K\big[\big[t^{1/N},\underline{x}\big]\big]$ then it defines the germ of a space curve through the origin. This allows us then in Corollary 3.6 to apply normalisation to find a negative-valued point in the tropical variety of $\gamma_{\omega,u}(J)$.

Lemma 3.5

Let $J \lhd L[\underline{x}]$, let $\omega \in \mathrm{Trop}(J)\cap\frac{1}{N}\cdot\mathbb{Z}^n$, and $u \in V\big(\mathrm{t}-\mathrm{in}_\omega(J)\big) \subset K^n$. Then

$$\gamma_{\omega,u}(J)\cap R_N[\underline{x}] \subseteq \langle t^{1/N}, x_1,\ldots,x_n\rangle \lhd R_N[\underline{x}].$$

Proof. Let $w = (-1,\omega)$ and $0 \neq f = \gamma_{\omega,u}(h) \in \gamma_{\omega,u}(J)\cap R_N[\underline{x}]$ with $h \in J$. Since f is a polynomial in $\underline{x}$ we have

$$h = \gamma_{\omega,u}^{-1}(f) = f(t^{\omega_1}\cdot x_1 - u_1,\ldots,t^{\omega_n}\cdot x_n - u_n) \in t^m\cdot R_N[\underline{x}]$$

for some $m \in \frac{1}{N}\cdot\mathbb{Z}$. We can thus decompose $g := t^{-m}\cdot h \in J_{R_N}$ into its w-quasihomogeneous parts, say

$$t^{-m}\cdot h = g = \sum_{q\le\hat{q}} g_{q,w},$$

where $\hat{q} = \mathrm{ord}_\omega(g)$ and thus $g_{\hat{q},w} = \mathrm{in}_\omega(g)$ is the w-initial form of g. As we have seen in Remark 3.4 there are polynomials $p_{g_{q,w},u} \in \langle x_1,\ldots,x_n\rangle \lhd K[\underline{x}]$ such that

$$\gamma_{\omega,u}(g_{q,w}) = t^{-q}\cdot g_{q,w}(1,u) + t^{-q}\cdot p_{g_{q,w},u}.$$

But then

$$\begin{aligned} f &= \gamma_{\omega,u}(h) = \gamma_{\omega,u}(t^m \cdot g) = t^m \cdot \gamma_{\omega,u}(g) = t^m \cdot \gamma_{\omega,u}\left(\sum_{q\leq\hat{q}} g_{q,\omega}\right) \\ &= t^m \cdot \sum_{q\leq\hat{q}} \left(t^{-q} \cdot g_{q,w}(1,u) + t^{-q} \cdot p_{g_{q,w},u}\right) \\ &= t^{m-\hat{q}} \cdot g_{\hat{q},w}(1,u) + t^{m-\hat{q}} \cdot p_{g_{\hat{q},w},u} + \sum_{q<\hat{q}} t^{m-q} \cdot \big(g_{q,w}(1,u) + p_{g_{q,w},u}\big). \end{aligned}$$

However, since $g \in J$ and $u \in V\big(\mathrm{t}-\mathrm{in}_\omega(J)\big)$ we have

$$g_{\hat{q},w}(1,u) = \mathrm{t}-\mathrm{in}_\omega(g)(u) = 0$$

and thus using (1) we get

$$p_{g_{\hat{q},w},u} = t^{\hat{q}} \cdot \Big(\gamma_{\omega,u}(g_{\hat{q},w}) - t^{-\hat{q}} \cdot g_{\hat{q},w}(1,u)\Big) = t^{\hat{q}} \cdot \gamma_{\omega,u}(g_{\hat{q},w}) \neq 0,$$

since $g_{\hat{q},w} = \mathrm{in}_\omega(g) \neq 0$ and $\gamma_{\omega,u}$ is an isomorphism. We see in particular, that $m-\hat{q} \geq 0$ since $f \in R_N[\underline{x}]$ and $p_{g_{\hat{q},w},u} \in \langle x_1,\ldots,x_n\rangle \lhd K[\underline{x}]$, and hence

$$f = t^{m-\hat{q}} \cdot p_{g_{\hat{q},w},u} + \sum_{q<\hat{q}} t^{m-q} \cdot \big(g_{q,w}(1,u) + p_{g_{q,w},u}\big) \in \langle t^{1/N}, x_1,\ldots,x_n\rangle. \qquad \square$$

The following corollary assures the existence of a negative-valued point in the tropical variety of the transformed ideal – after possibly eliminating those variables for which the components of the solution will be zero.

Corollary 3.6

Suppose that K is an algebraically closed field of characteristic zero. Let $J \lhd L[\underline{x}]$ be a zero-dimensional ideal, let $\omega \in \mathrm{Trop}(J)\cap\mathbb{Q}^n$, and $u \in V\big(\mathrm{t}-\mathrm{in}_\omega(J)\big) \subset K^n$. Then

$$\exists\, p = (p_1,\ldots,p_n) \in V\big(\gamma_{\omega,u}(J)\big)\ :\ \forall i:\ \mathrm{val}(p_i) \in \mathbb{Q}_{>0}\cup\{\infty\}.$$

In particular, if $n_p = \#\{p_i \mid p_i \neq 0\} > 0$ and $\underline{x}_p = (x_i \mid p_i \neq 0)$, then

$$\mathrm{Trop}\big(\gamma_{\omega,u}(J)\cap L[\underline{x}_p]\big)\cap\mathbb{Q}^{n_p}_{<0} \neq \emptyset.$$

Proof. We may choose an $N \in \mathcal{N}(\gamma_{\omega,u}(J))$ and such that $\omega \in \frac{1}{N}\cdot\mathbb{Z}^n_{\leq 0}$. Let $I = \gamma_{\omega,u}(J)\cap R_N[\underline{x}]$.

Since $\gamma_{\omega,u}$ is an isomorphism we know that

$$0 = \dim(J) = \dim\big(\gamma_{\omega,u}(J)\big),$$

and by Proposition 5.3 we know that

$$\mathrm{Ass}(I) = \big\{P_{R_N} \mid P \in \mathrm{Ass}\big(\gamma_{\omega,u}(J)\big)\big\}.$$

Since the maximal ideal

$$\mathfrak{m} = \langle t^{1/N}, x_1,\ldots,x_n\rangle_{R_N[\underline{x}]} \lhd R_N[\underline{x}]$$

contains the element $t^{1/N}$, which is a unit in $L[\underline{x}]$, it cannot be the contraction of a prime ideal in $L[\underline{x}]$. In particular, $\mathfrak{m} \notin \mathrm{Ass}(I)$. Thus there must be a $P \in \mathrm{Ass}(I)$ such that $P \subsetneqq \mathfrak{m}$, since by Lemma 3.5 $I \subset \mathfrak{m}$ and since otherwise $\mathfrak{m}$ would be minimal over I and hence associated to I.

The strict inclusion implies that $\dim(P) \geq 1$, while Theorem 6.10 shows that

$$\dim(P) \leq \dim(I) \leq \dim\big(\gamma_{\omega,u}(J)\big) + 1 = 1.$$

Hence the ideal P is a 1-dimensional prime ideal in $R_N[\underline{x}] \subset K\big[[t^{1/N}, \underline{x}]\big]$, where the latter is the completion of the former with respect to $\mathfrak{m}$. Since $P \subset \mathfrak{m}$, the completion $\hat{P}$ of P with respect to $\mathfrak{m}$ is also 1-dimensional and the normalisation

$$\psi : K\big[[t^{1/N}, \underline{x}]\big]/\hat{P} \hookrightarrow \widetilde{R} \simeq K[[s]]$$

gives a parametrisation where we may assume that $\psi(t^{1/N}) = s^M$ for some $M \in \mathbb{N}_{>0}$ since K is algebraically closed and of characteristic zero (see e.g. [4, Corollary 4.4.10] for $K = \mathbb{C}$). Let now $s_i = \psi(x_i) \in K[[s]]$ then necessarily $a_i = \mathrm{ord}_s(s_i) > 0$, since ψ is a local K-algebra homomorphism, and $f(s^M, s_1, \ldots, s_n) = \psi(f) = 0$ for all $f \in \hat{P}$. Taking $I \subseteq P \subset \hat{P}$ and $\gamma_{\omega,u}(J) = \langle I \rangle$ into account and replacing s by $t^{1/(N\cdot M)}$ we get

$$f(t^{1/N}, p) = 0 \quad \text{for all} \;\; f \in \gamma_{\omega,u}(J)$$

where

$$p = \Big(s_1\big(t^{1/(N\cdot M)}\big), \ldots, s_n\big(t^{1/(N\cdot M)}\big)\Big) \in R^n_{N\cdot M} \subseteq L^n.$$

Moreover,

$$\mathrm{val}(p_i) = \frac{a_i}{N \cdot M} \in \mathbb{Q}_{>0} \cup \{\infty\},$$

and every $f \in \gamma_{\omega,u}(J) \cap L[\underline{x}_p]$ vanishes at the point $p' = (p_i \mid p_i \neq 0)$. By Proposition 2.14

$$-\mathrm{val}(p') \in \mathrm{Trop}\big(\gamma_{\omega,u}(J) \cap L[\underline{x}_p]\big) \cap \mathbb{Q}^{n_p}_{<0}. \qquad \square$$

Constructive Proof of Theorem 3.1 Recall that by Remark 3.3 we may assume that $\omega \in \mathbb{Q}^n_{<0}$. It is our first aim to construct recursively sequences of the following objects for $\nu \in \mathbb{N}$:

- natural numbers $1 \leq n_\nu \leq n$,
- natural numbers $1 \leq i_{\nu,1} < \ldots < i_{\nu,n_\nu} \leq n$,
- subsets of variables $\underline{x}_\nu = (x_{i_{\nu,1}}, \ldots, x_{i_{\nu,n_\nu}})$,
- ideals $J'_\nu \lhd L[\underline{x}_{\nu-1}]$,
- ideals $J_\nu \lhd L[\underline{x}_\nu]$,
- vectors $\omega_\nu = (\omega_{\nu,i_{\nu,1}}, \ldots, \omega_{\nu,i_{\nu,n_\nu}}) \in \mathrm{Trop}(J_\nu) \cap (\mathbb{Q}_{<0})^{n_\nu}$, and
- vectors $u_\nu = (u_{\nu,i_{\nu,1}}, \ldots, u_{\nu,i_{\nu,n_\nu}}) \in V\big(\mathrm{t}-\mathrm{in}_{\omega_\nu}(J_\nu)\big) \cap (K^*)^{n_\nu}$.

We set $n_0 = n$, $\underline{x}_{-1} = \underline{x}_0 = \underline{x}$, $J_0 = J'_0 = J$, and $\omega_0 = \omega$, and since $\mathrm{t}-\mathrm{in}_\omega(J)$ is monomial free by assumption and K is algebraically closed we may choose a $u_0 \in V\big(\mathrm{t}-\mathrm{in}_{\omega_0}(J_0)\big) \cap (K^*)^{n_0}$. We then define recursively for $\nu \geq 1$

$$J'_\nu = \gamma_{\omega_{\nu-1},u_{\nu-1}}(J_{\nu-1}).$$

By Corollary 3.6 we may choose a point $q \in V(J'_\nu) \subset L^{n_{\nu-1}}$ such that $\mathrm{val}(q_i) = \mathrm{ord}_t(q_i) > 0$ for all $i = 1, \ldots, n_{\nu-1}$. As in Corollary 3.6 we set

$$n_\nu = \#\{q_i \mid q_i \neq 0\} \in \{0, \ldots, n_{\nu-1}\},$$

and we denote by

$$1 \leq i_{\nu,1} < \ldots < i_{\nu,n_\nu} \leq n$$

the indexes i such that $q_i \neq 0$.

If $n_\nu = 0$ we simply stop the process, while if $n_\nu \neq 0$ we set

$$\underline{x}_\nu = (x_{i_{\nu,1}}, \ldots, x_{i_{\nu,n_\nu}}) \subseteq \underline{x}_{\nu-1}.$$

We then set

$$J_\nu = \left(J'_\nu + \langle \underline{x}_{\nu-1} \setminus \underline{x}_\nu \rangle\right) \cap L[\underline{x}_\nu],$$

and by Corollary 3.6 we can choose

$$\omega_\nu = (\omega_{\nu,i_{\nu,1}}, \ldots, \omega_{\nu,i_{\nu,n_\nu}}) \in \mathrm{Trop}(J_\nu) \cap \mathbb{Q}^{n_\nu}_{<0}.$$

Then $\mathrm{t}-\mathrm{in}_{\omega_\nu}(J_\nu)$ is monomial free, so that we can choose a

$$u_\nu = (u_{\nu,i_{\nu,1}}, \ldots, u_{\nu,i_{\nu,n_\nu}}) \in V\big(\mathrm{t}-\mathrm{in}_{\omega_\nu}(J_\nu)\big) \cap (K^*)^{n_\nu}.$$

Next we define

$$\varepsilon_i = \sup\big\{\nu \bigm| i \in \{i_{\nu,1}, \ldots, i_{\nu,n_\nu}\}\big\} \in \mathbb{N} \cup \{\infty\} \text{ and}$$

$$p_{\mu,i} = \sum_{\nu=0}^{\min\{\varepsilon_i,\mu\}} u_{\nu,i} \cdot t^{-\sum_{j=0}^{\nu} \omega_{j,i}}$$

for $i = 1, \ldots, n$. All $\omega_{\nu,i}$ are strictly negative, which is necessary to see that the $p_{\mu,i}$ converge to a Puiseux series. Note that in the case $n = 1$ the described procedure is just the classical Puiseux expansion (see e.g. [4, Theorem 5.1.1] for the case $K = \mathbb{C}$). To see that the $p_{\mu,i}$ converge to a Puiseux series (i.e. that there exists a common denominator N for the exponents as μ goes to infinity), the general case can easily be reduced to the case $n = 1$ by projecting the variety to all coordinate lines, analogously to the [17, proof in Section 3]. The ideal of the projection to one coordinate line is principal. Transformation and intersection commute.

It is also easy to see that at $p = (p_1, \ldots, p_n) \in L^n$ all polynomials in J vanish, where

$$p_i = \lim_{\mu \to \infty} p_{\mu,i} = \sum_{\nu=0}^{\infty} u_{\nu,i} \cdot t^{-\sum_{j=0}^{\nu} \omega_{j,i}} \in R_N \subset L.$$

□

Remark 3.7 The proof is basically an algorithm which allows to compute a point $p \in V(J)$ such that $\mathrm{val}(p) = -\omega$. However, if we want to use a computer algebra system like Singular for the computations, then we have to restrict to generators of J which are polynomials in $t^{1/N}$ as well as in $\underline{x}$. Moreover, we should pass from $t^{1/N}$ to t, which can be easily done by the K-algebra isomorphism

$$\Psi_N : L[\underline{x}] \longrightarrow L[\underline{x}] : t \mapsto t^N, x_i \mapsto x_i.$$

Whenever we do a transformation which involves rational exponents we will clear the denominators using this map with an appropriate N.

We will in the course of the algorithm have to compute the t-initial ideal of J with respect to some $\omega \in \mathbb{Q}^n$, and we will do so by a standard basis computation using the monomial ordering $>_\omega$, given by

$$t^\alpha \cdot \underline{x}^\beta \;>_\omega\; t^{\alpha'} \cdot \underline{x}^{\beta'} \iff$$
$$-\alpha + \omega \cdot \beta > -\alpha' + \omega \cdot \beta' \text{ or } (-\alpha + \omega \cdot \beta = -\alpha' + \omega \cdot \beta' \text{ and } \underline{x}^\beta > \underline{x}^{\beta'}),$$

where $>$ is some fixed global monomial ordering on the monomials in $\underline{x}$.

Algorithm 3.8 (ZDL – Zero dimensional lifting algorithm)

INPUT: $(m, f_1, \ldots, f_k, \omega) \in \mathbb{N}_{>0} \times K[t, \underline{x}]^k \times \mathbb{Q}^n$ such that $\dim(J) = 0$ and $\omega \in \mathrm{Trop}(J)$ for $J = \langle f_1, \ldots, f_k \rangle_{L[\underline{x}]}$.

OUTPUT: $(N, p) \in \mathbb{N} \times K[t, t^{-1}]^n$ such that $p(t^{1/N})$ coincides with the first m terms of a solution of $V(J)$ and such that $\mathrm{val}(p) = -\omega$.

INSTRUCTIONS:

- Choose $N \geq 1$ such that $N \cdot \omega \in \mathbb{Z}^n$.
- FOR $i = 1, \ldots, k$ DO $f_i := \Psi_N(f_i)$.
- $\omega := N \cdot \omega$
- IF some $\omega_i > 0$ THEN
  - FOR $i = 1, \ldots, k$ DO $f_i := \Phi_\omega(f_i) \cdot t^{-\mathrm{ord}_t\left(\Phi_\omega(f_i)\right)}$.
  - $\tilde{\omega} := \omega$.
  - $\omega := (0, \ldots, 0)$.
- Compute a standard basis $(g_1, \ldots, g_l)$ of $\langle f_1, \ldots, f_k \rangle_{K[t,\underline{x}]}$ with respect to the ordering $>_\omega$.
- Compute a zero $u \in (K^*)^n$ of $\langle \mathrm{t} - \mathrm{in}_\omega(g_1), \ldots, \mathrm{t} - \mathrm{in}_\omega(g_l) \rangle_{K[\underline{x}]}$.
- IF $m = 1$ THEN $(N, p) := \left(N, u_1 \cdot t^{-\omega_1}, \ldots, u_n \cdot t^{-\omega_n}\right)$.
- ELSE
  - Set $G = \left(\gamma_{\omega,u}(f_i) \;\middle|\; i = 1, \ldots, k\right)$.
  - FOR $i = 1, \ldots, n$ DO
    * Compute a generating set G' of $\langle G, x_i \rangle_{K[t,\underline{x}]} : \langle t \rangle^\infty$.
    * IF $G' \subseteq \langle t, \underline{x} \rangle$ THEN
      · $\underline{x} := \underline{x} \setminus \{x_i\}$
      · Replace G by a generating set of $\langle G' \rangle \cap K[t, \underline{x}]$.
  - IF $\underline{x} = \emptyset$ THEN $(N, p) := \left(N, u_1 \cdot t^{-\omega_1}, \ldots, u_n \cdot t^{-\omega_n}\right)$.
  - ELSE
    * Compute a point ω' in the negative orthant of the tropical variety of $\langle G \rangle_{L[\underline{x}]}$.
    * $(N', p') = ZDL(m - 1, G, \omega')$.
    * $N := N \cdot N'$.
    * FOR $j = 1, \ldots, n$ DO
      · IF $x_i \in \underline{x}$ THEN $p_i := t^{-\omega_i \cdot N'} \cdot (u_i + p'_i)$.
      · ELSE $p_i := t^{-\omega_i \cdot N'} \cdot u_i$.

- IF some $\tilde{\omega}_i > 0$ THEN $p := \big(t^{-\tilde{\omega}_1} \cdot p_1, \dots, t^{-\tilde{\omega}_n} \cdot p_n\big)$.

Proof. The algorithm which we describe here is basically one recursion step in the constructive proof of Theorem 3.1 given above, and thus the correctness follows once we have justified why our computations do what is required by the recursion step. Notice that Step 4 and the last step make an adjusting change of variables to make all ω_i non-positive in the body of the algorithm. This together with Step 3 guarantees that $t^{-\omega_i}$ is a polynomial.

If we compute a standard basis $(g_1, \dots, g_l)$ of $\langle f_1, \dots, f_k\rangle_{K[t,\underline{x}]}$ with respect to $>_\omega$, then by Theorem 2.8 the t-initial forms of the g_i generate the t-initial ideal of $J = \langle f_1, \dots, f_k\rangle_{L[\underline{x}]}$. We thus compute a zero u of the t-initial ideal as required.

Next the recursion in the proof of Theorem 3.1 requires to find an $\omega \in \big(\mathbb{Q}_{>0} \cup \{\infty\}\big)^n$, which is $-\mathrm{val}(q)$ for some $q \in V(J)$, and we have to eliminate those components which are zero. Note that the solutions with first component zero are the solutions of $J + \langle x_1\rangle$. Checking if there is a solution with strictly positive valuation amounts by the proof of Corollary 3.6 to checking if $\big(J + \langle x_1\rangle\big) \cap K[[t]][\underline{x}] \subseteq \langle t, \underline{x}\rangle$, and the latter is equivalent to $G' \subseteq \langle t, \underline{x}\rangle$ by Lemma 3.9. If so, we eliminate the variable x_1 from $\langle G'\rangle_{K[t,\underline{x}]}$, which amounts to projecting all solutions with first component zero to L^{n-1}. We then continue with the remaining variables. That way we find a set of variables $\{x_{i_1}, \dots, x_{i_s}\}$ such that there is a solution of $V(J)$ with strictly positive valuation where precisely the other components are zero.

The rest follows from the constructive proof of Theorem 3.1. $\square$

Lemma 3.9

Let $f_1, \dots, f_k \in K[t, \underline{x}]$, $J = \langle f_1, \dots, f_k\rangle_{L[\underline{x}]}$, $I = \langle f_1, \dots, f_k\rangle_{K[t,\underline{x}]} : \langle t\rangle^\infty$, and let G be a generating set of I. Then:

$$J \cap K[[t]][\underline{x}] \subseteq \langle t, \underline{x}\rangle \iff I \subseteq \langle t, \underline{x}\rangle \iff G \subseteq \langle t, \underline{x}\rangle.$$

Proof. The last equivalence is clear since I is generated by G, and for the first equivalence it suffices to show that $J \cap K[[t]][\underline{x}] = \langle I\rangle_{K[[t]][\underline{x}]}$.

For this let us consider the following two ideals $I' = \langle f_1, \dots, f_k\rangle_{K[[t]][\underline{x}]} : \langle t\rangle^\infty$ and $I'' = \langle f_1, \dots, f_k\rangle_{K[t]_{\langle t\rangle}[\underline{x}]} : \langle t\rangle^\infty$. By Lemma 6.6 we know that $J \cap K[[t]][\underline{x}] = I'$ and by [15, Proposition 6.20] we know that $I' = \langle I''\rangle_{K[[t]][\underline{x}]}$. It thus suffice to show that $I'' = \langle I\rangle_{K[t]_{\langle t\rangle}[\underline{x}]}$. Obviously $I \subseteq I''$, which proves one inclusion. Conversely, if $f \in I''$ then f satisfies a relation of the form

$$t^m \cdot f \cdot u = \sum_{i=1}^k g_i \cdot f_i,$$

with $m \geq 0$, $u \in K[t]$, $u(0) = 1$ and $g_1, \dots, g_k \in K[t, \underline{x}]$. Thus $f \cdot u \in I$ and $f = \frac{f \cdot u}{u} \in \langle I\rangle_{K[t]_{\langle t\rangle}[\underline{x}]}$. $\square$

Remark 3.10 In order to compute the point ω' we may want to compute the tropical variety of $\langle G\rangle_{L[\underline{x}]}$. The tropical variety can be computed as a subcomplex of a Gröbner fan or more efficiently by applying [3, Algorithm 5] for computing tropical bases of tropical curves.