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Professor Joan Cerdà has served the last twenty-one years as editor of Collectanea Mathematica and Javier Soria has served the last fifteen years. This has been a period of changes and challenges, among them the full digitization of the journal. We have benefited very much from their work and clearly there has been a marked improvement in the quality of the journal during their term.

It is with a grateful acknowledgment to them that we assume the task of editing Collectanea Mathematica.

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Smoothing effect for regularized Schrödinger equation on compact manifolds

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Abstract

We prove smoothing effect for solutions of a regularized Schrödinger equation on compact manifolds under the hypothesis of the Geometric control.

1. Introduction

It is well known that the solution of the free Schrödinger equation

$$\begin{cases} i\partial_t u - \Delta u = 0 & \text{in} \quad \mathbb{R}\times\mathbb{R}^d \\ u(0,.) = u_0 & \text{in} \quad \mathbb{R}^d, \end{cases} \tag{1}$$

satisfies the following smoothing effect

$$(i) \qquad \int_{\mathbb{R}}\int_{|x|<R} |(1-\Delta)^{1/4}u(t,x)|^2 dxdt \le c_R \left\|u_0\right\|_{L^2(\mathbb{R}^d)},$$

for all $u_0 \in L^2(\mathbb{R}^d)$ *and all nonnegative* R.

$$(ii) \qquad u \in C^\infty(\mathbb{R}\setminus\{0\}\times\mathbb{R}^d),$$

for all $u_0 \in L^2(\mathbb{R}^d)$ *with a compact support.*

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The property (i) implies that $u(t,x)$ belongs, for almost all t, to the local Sobolev space $H^{1/2}_{\text{loc}}(\mathbb{R}^d)$. It has been, first, observed by Sjölin [18], Vega [19] and Constantin-Saut [7] and it was later generalized to different perturbations of the flat Laplacian (see Ben Artzi-Klainerman [4], Ben Artzi-Devinatz [3] Constantin-Saut [8] and Doï [11, 12]).

In the case of exterior non-trapping domain, Burq, Gerard and Tzvetkov [5] proved the smoothing effect for Schrödinger equation with Dirichlet condition. This result was proved in [6] with loss of ε- derivatives for some captive geometry.

Doï [10] has proved that, for Schrödinger operators in $\mathbb{R}^d$, the non trapping assumption is necessary for the $H^{1/2}$ smoothing effect. This result has been extended to the case of boundary value problems by Burq [6].

As a conclusion, we can assert that the smoothing property for Schrödinger equation is equivalent to the non trapping condition.

Furthermore, what concerns the wave equation, this non trapping condition has also a relationship with the problem of the uniform local energy decay. Indeed this decay is equivalent to the non trapping hypothesis (see [14] for the necessary condition and [13] for the sufficiency).

In the captive case, the stabilization of the wave equation was achieved by adding a dissipative term of type $a(x)\partial_t u$. Under a microlocal geometric assumption called "geometric control", J. Rauch and M. Taylor [16] proved the uniform energy decay for the dissipative wave equation on compact manifold (where all rays are captive). This result was extended to the case of manifolds with boundary by C. Bardos, G. Lebeau and J. Rauch [2] (see [1] for exterior domain).

By analogy with the stabilization problem for the wave equation we are interested, in this paper, in producing the smoothing effect "by force" for Schrödinger equation on compact manifolds by adding to the equation a regularizing term $ia(x)(1-\Delta)^\alpha a(x)u$, where a is a real valued smooth function and $0 < \alpha \leq \frac{1}{2}$.

More precisely, when M is a compact connected Riemannian manifold of dimension $d \geq 2$ and Δ the corresponding Laplace-Beltrami operator, we consider the following regularized Schrödinger equation

$$\begin{cases} i\partial_t u - \Delta u + ia(x)(1-\Delta)^\alpha a(x)u + R_0 u = 0 & \text{in} \quad \mathbb{R}_+ \times M \\ u(0,.) = u_0 & \text{in} \quad M. \end{cases} \tag{2}$$

Here $0 < \alpha \leq \frac{1}{2}$, $a(x) \in C^\infty(M)$ and R_0 is a pseudo-differential operator of order zero.

In Section 2 we will prove that the problem (2) is well posed in $H^s(M)$ for all $s \in \mathbb{R}$ and we will denote by $(U(t))_{t\geq 0}$ the semi group which generates its solution: $u(t,.) = U(t)u_0$.

The aim of this article is to establish, under the "geometric control" condition, some smoothing effects for equation (2). More precisely we have.

Theorem 1

Assume that $\omega = \{a(x) \neq 0\}$ controls geometrically M, i.e every geodesic of M enters the set ω. Then the following hold:

(a) *For $T > 0$ and $f \in L^2([0,T], H^s(M))$, we have*

$$u \in L^2_{\text{loc}}((0,T), H^{s+\alpha}(M)) \tag{3}$$

where $u(t,.) = \int_0^t U(t-\tau)f(\tau,.)d\tau, (t \geq 0)$.

Furthermore, for every $\theta \in C_0^\infty((0,T))$, there exists $c > 0$ such that

$$\|\theta u\|_{L^2(\mathbb{R},H^{s+\alpha}(M))} \leq c\,\|f\|_{L^2([0,T],H^s(M))}\,. \tag{4}$$

(b) *Let $v_0 \in H^s(M)$, we have*

$$v \in C^\infty((0,+\infty) \times M) \tag{5}$$

where v is the solution of (2) with initial data v_0.

Furthermore, for every $\varphi \in C_0^\infty((0,+\infty))$, $\beta \in \mathbb{N}^{d+1}$ there exists $c > 0$ such that

$$\sup_{(t,x)\in\mathbb{R}\times M} |D^\beta(\varphi v)(t,x)| \leq c\,\|v_0\|_{H^s(M)} \tag{6}$$

Remark 2 Taking into account that our problem is on compact manifold, any gain of δ-regularity ($\delta > 0$) with respect to initial data leads by iteration to C^∞ smoothing effect. That is to say that, outside any neighborhood of $t = 0$, our result is still the same as the (i) type of smoothing estimate, for which the nontrapping condition is necessary. Therefore, we can say that the necessity of our geometric control condition is not clear; this constitutes an open problem.

We conclude this introduction with a short description of Proof of Theorem 1. We first check that if $u_0 \in H^s(M)$ and $f \in L^2([0,T]; H^s(M))$ then the function

$$u = U(t)u_0 + \int_0^t U(t-\tau)f(\tau,.)d\tau \in L^2([0,T]; H^{s+\alpha}(\omega))$$

(Lemma 1). Then, we prove that the $L^2_{\text{loc}}((0,T); H^{s+\alpha})$ regularity propagates over the bicaracteristic flow of Δ for solutions of (2) (Proposition 1). The proof of this result is very similar to the one of [9]. Using the propagation of regularity we deduce that $u \in L^2_{\text{loc}}((0,T); H^{s+\alpha}(M))$ which yields (3). Applying the closed graph theorem, we obtain the estimate (4) (Proposition 3). The proof of the C^∞ smoothing effect (5) is completed by iteration of the previous regularity result for $f = 0$. Finally, using, once again, the closed graph theorem and the Sobolev embedding, we deduce the estimate (6).

The rest of this article is organized as follows:
2) Well-posedness.
3) Propagation of regularity.
4) Proof of Theorem 1.

2. Well-posedness

In this section we establish that the Cauchy problem (2) is well posed in $H^s(M)$ ($s \in \mathbb{R}$).

We denote by $A_a = -i\Delta - a(x)(1-\Delta)^\alpha a(x) + iR_0$ the unbounded operator defined on $L^2(M)$ with domain $D(A_a) = \{f \in L^2(M); A_a f \in L^2(M)\}$ and set $B_a = a(x)(1-\Delta)^\alpha a(x)$.

Proposition 1

The problem (2) is well posed in $H^s(M)$ ($s \in \mathbb{R}$).

Proof. We begin by proving the proposition for $s = 0$. Let $f \in D(A_a)$, we have

$$\begin{aligned} Re(A_a f, f) &= -\|(1-\Delta)^{\alpha/2} a(x) f(x)\|^2_{L^2(M)} + Im(R_0 f, f) \\ &\leq c\|f\|^2_{L^2(M)}, \end{aligned}$$

so $A_a - c$ is dissipative. It remains to show that the operator $A_a - c$ is maximal.

Let $g \in L^2(M)$, we look for $f \in D(A_a)$ satisfying $f - A_a f + cf = g$, i.e.

$$i\Delta f + a(x)(1-\Delta)^\alpha a(x) f + (1+c) f = g \qquad \text{in } M.$$

Let us define the bilinear form

$$\begin{aligned} b:\ H^1(M) \times H^1(M) &\longrightarrow \mathbb{C} \\ (f,h) &\longmapsto \int_M (-i\nabla f \overline{\nabla h} + (1+c) f\bar{h} + (1-\Delta)^{\alpha/2} a f \overline{(1-\Delta)^{\alpha/2} a h}) dx \end{aligned}$$

and the linear form

$$\begin{aligned} L:\ H^1(M) &\longrightarrow \mathbb{C} \\ h &\longmapsto \int g\bar{h}. \end{aligned}$$

It is easy to see that b and L are continuous and that

$$\forall f \in H^1(M), \qquad |b(f,f)| \geq \|f\|^2_{H^1}.$$

Hence b is coercive. According to the Lax Milgram theorem, there exists a unique $f \in H^1(M)$ such that

$$b(f,h) = L(h), \forall h \in H^1(M).$$

We deduce that

$$i\Delta f + a(x)(1-\Delta)^\alpha a(x) f + (1+c) f = g \ \text{ in } \ \mathcal{D}'(M)$$

then $\Delta f \in L^2(M)$ and so $f \in D(A_a)$.

The operator $A_a - c$ is maximal and dissipative, therefore according to the Hille-Yoshida theorem [17], A_a generates a semi group $(U(t))_{t\geq 0}$ such that if $f \in L^2(M)$ then $U(t)f \in C([0,+\infty), L^2(M))$ is the unique solution of (2).

Now, we prove the well posedness of problem (2) in $H^s(M)$, $s \in \mathbb{R}$.

Let $u_0 \in H^s(M)$, set $v_0 = (1-\Delta)^{s/2}u_0 \in L^2(M)$ and consider the following problem

$$\begin{cases} i\partial_t v - \Delta v + iB_a v + G_0 v = 0 & \text{in} \quad \mathbb{R}_+ \times M \\ v(0,.) = v_0 & \text{in} \quad M, \end{cases} \tag{7}$$

where

$$G_0 = R_0 + [(1-\Delta)^{s/2}, iB_a + R_0](1-\Delta)^{-s/2},$$

is a pseudo-differential of order zero. Taking into account what precedes, problem (7) admits a unique solution $v \in C(\mathbb{R}_+, L^2(M))$. Set $u = (1-\Delta)^{-s/2}v \in C(\mathbb{R}_+, H^s(M))$, it is easy to see that u is the unique solution of the problem (2). $\square$

3. Propagation of regularity

We are interested, in this section, in the regularity of solutions of the following problem

$$\begin{cases} i\partial_t u - \Delta u + ia(x)(1-\Delta)^{\alpha}a(x)u = f \\ u(0,.) = u_0, \end{cases} \tag{8}$$

where $0 \le \alpha \le \dfrac{1}{2}$.

The following lemma shows that the new term $(ia(x)(1-\Delta)^{\alpha}a(x)u)$ goes in the "right way", in the sense that it guarantees the smoothness of solutions of (8) over the regularizing set $\omega = \{x \in M,\ a(x) \neq 0\}$.

Lemma 1

Let $u_0 \in H^s(M)$, $s \in \mathbb{R}$ and $f \in L^2([0,T], H^s(M))$, if u is a solution of (8), then $au \in L^2([0,T], H^{s+\alpha}(M))$. Furthermore, there exists $c > 0$ such that

$$\int_0^T \|au\|^2_{H^{s+\alpha}(M)} \le c\Big(\|u_0\|^2_{H^s(M)} + \|f\|^2_{L^2([0,T],H^s(M))}\Big).$$

Proof. We multiply (8) by $(1-\Delta)^s\bar{u}$, we integrate over $[0,T]\times M$ and take the imaginary part. We obtain

$$\begin{aligned} \int_0^T \|au\|^2_{H^{s+\alpha}(M)} &= \frac{1}{2}\|u(0,.)\|^2_{H^s(M)} - \frac{1}{2}\|u(T,.)\|^2_{H^s(M)} \\ &- Re\Big(\int_0^T ((1-\Delta)^{s/2}a(x)u, (1-\Delta)^{-s/2+\alpha}[a,(1-\Delta)^s]u)_{L^2(M)}\Big) \\ &+ Im\Big(\int_0^T (f,(1-\Delta)^s u)_{L^2(M)}\Big). \end{aligned}$$

The notation Re and Im stand for the real and imaginary part respectively. Since $u_0 \in H^s(M)$, $u \in C([0,T], H^s(M))$ and then

$$\Big|\int_0^T ((1-\Delta)^{s/2}a(x)u, (1-\Delta)^{-s/2+\alpha}[a,(1-\Delta)^s]u)_{L^2(M)}\Big| \le c\|u_0\|^2_{H^s(M)}.$$

We have also $f \in L^2([0,T], H^s(M))$, so

$$\begin{aligned}\Big|\int_0^T (f,(1-\Delta)^s u)_{L^2(M)}\Big| &\le c\|u_0\|_{H^s(M)}\|f\|_{L^2([0,T],H^s(M))}\\ &\le c\big(\|u_0\|^2_{H^s(M)} + \|f\|^2_{L^2([0,T],H^s(M))}\big).\end{aligned}$$

Consequently

$$\int_0^T \|au\|^2_{H^{s+\alpha}(M)} \le c\big(\|u_0\|^2_{H^s(M)} + \|f\|^2_{L^2([0,T],H^s(M))}\big).$$

This completes the Proof of Lemma 1. □

Now, let us define the microlocal regularity needed in the next proposition.

DEFINITION 1 Let $\rho \in T^\star M \setminus \{0\}$, $s \in \mathbb{R}$ and $T > 0$, we say that $u \in L^2_{\mathrm{loc}}((0,T), H^s(\rho))$ if there exists a 0-order pseudo-differential operator $\chi(x, D_x)$, elliptic in ρ, such that

$$\chi(x, D_x)u \in L^2_{\mathrm{loc}}((0,T), H^s(M)).$$

Moreover we denote by Γ_ρ the geodesic ray starting at ρ.

The key point in the Proof of Theorem 1 is the following propagation of regularity.

Proposition 2

With the notation above, let u be a solution of (8) with $u_0 \in H^s(M)$ and $f \in L^2([0,T], H^s(M))$. Let $\rho_0 \in T^\star M \setminus \{0\}$ and assume that

$$u \in L^2_{\mathrm{loc}}((0,T), H^{s+\alpha}(\rho_0)).$$

Then, for every $\rho_1 \in \Gamma_{\rho_0}$, we have

$$u \in L^2_{\mathrm{loc}}((0,T), H^{s+\alpha}(\rho_1)).$$

For the proof of this proposition we need the following lemma.

Lemma 2

Let $\rho_0 \in T^\star M \setminus \{0\}$. There exists a conical neighborhood U_1 of ρ_0 satisfying the following:

For every conical neighborhood $U_2 \subset\subset U_1$ of ρ_0 and $k(x,\xi)$ a symbol of order ν supported in U_2, there exists a unique symbol $q(x,\xi)$ on U_1 of order $\nu - 1$ satisfying

$$\begin{cases} H_p q = k \\ \operatorname{supp}(q) \subset \cup_{\tau \ge 0}\Phi_\tau(U_2) \end{cases} \tag{9}$$

where H_p is the Hamiltonian vector field associated to $p(x,\xi) = |\xi|^2$ and Φ_τ is the bicharacteristic flow of Δ.

Proof of Lemma 2. Set $\widetilde{p}(x,\xi) = |\xi|$. First, we can verify that the bicharacteristic curve of $H_{\widetilde{p}}$ starting from $\rho_0 = (x_0, \xi_0)$ coincides with that one of H_p starting from the same point. Second, we have $H_{\widetilde{p}} = \dfrac{\xi}{|\xi|}\partial_x$, then, setting $y = \Big(x, \dfrac{\xi}{|\xi|}\Big)$ and $t = |\xi|$, we can consider $H_{\widetilde{p}}$ as a vector field on $S^\star(M) \times \mathbb{R}_+^\star$, where $S^\star(M) = \{\xi \in T^\star M\ ,\ |\xi| = 1\}$. Since $\xi_0 \neq 0$, there exists a neighborhood V_1 of $y_0 = \Big(x_0, \dfrac{\xi_0}{|\xi_0|}\Big)$ and a C^∞diffeomophism defined on V_1 transforming y_0 to $(0,0)$ and the vector field $H_{\widetilde{p}}$ to ∂_{y_1}. Therefore, we try to solve the following differential equation

$$\begin{cases} \dfrac{\partial q}{\partial y_1} = k(y,t), & (y,t) \in U_1 \\ q = 0 & \text{for } y_1 \text{ negative enough} \end{cases} \tag{10}$$

where $U_1 = V_1 \times \mathbb{R}_+^\star$ is a conical neighborhood of ρ_0.

If we choose $V_2 \subset\subset V_1$ a neighborhood of $y_0 = 0$ and k a symbol of order ν supported in $U_2 = V_2 \times \mathbb{R}_+^\star$, then the solution q of (10) is a symbol on U_1 of order ν satisfying

$$\operatorname{supp}(q) \subset (\cup_{\tau \geq 0} \Phi_\tau(V_2)) \times \mathbb{R}_+^\star = \cup_{\tau \geq 0} \Phi_\tau(U_2).$$

Finally, it is easy to see that the function $\dfrac{q}{|\xi|}$ satisfies (9). $\square$

Proof of Proposition 2. The proof is similar to that of [9]. In fact, we will just show that one can reproduce it by adding the regularizing term $ia(x)(1-\Delta)^\alpha a(x)u$.

We argue by contradiction. Suppose that the set of $t > 0$ such that $u \notin L^2_{\text{loc}}((0,T), H^{s+\alpha}(\Phi_t(\rho_0)))$ is not empty and denote by t_1 the infinimum bound of this set. It is clear that the set

$$\{\rho \in T^\star M \setminus \{0\};\ u \notin L^2_{\text{loc}}((0,T), H^{s+\alpha}(\rho))\}$$

is closed in $T^\star M \setminus \{0\}$, so $u \notin L^2_{\text{loc}}((0,T), H^{s+\alpha}(\rho_1 := \Phi_{t_1}(\rho_0)))$.

We will prove that $u \in L^2_{\text{loc}}((0,T), H^{s+\alpha}(\rho_1))$, which contradicts our assumption.

First we regularize u by taking $u_n = \Big(1 - \dfrac{1}{n^2}\Delta\Big)^{-1} u$. Consider U_1 a conical neighborhood of ρ_1 provided by Lemma 2 and choose $0 < t_2 < t_1$ and a small conical neighborhood W of $\rho_2 = \Phi_{t_2}(\rho_0)$ such that

$$\begin{cases} u \in L^2_{\text{loc}}((0,T), H^{s+\alpha}(\rho)) & \text{for all } \rho \in W \\ W \subset U_1. \end{cases} \tag{11}$$

Let $\delta > 0$ be small enough such that $\phi(\tau)\rho_0 \in W$ for $|\tau - t_2| < 2\delta$. Next, we fix a symbol χ of order zero supported in U_1 and verifying

- $\chi \equiv 1$ in a conical neighborhood of points $\phi(\tau)\rho_0$ for $\tau \in [t_2 + \delta, t_1]$,
- $\chi \equiv 0$ in a conical neighborhood of points $\phi(\tau)\rho_0$ for $\tau < t_2 + \dfrac{\delta}{2}$.

Then, we consider a small conical neighborhood U of ρ_1 such that

$$U \subset\subset U_1\ ,\quad \chi \equiv 1 \text{ in } U \quad \text{and} \quad (\cup_{\tau\le 0}\Phi_\tau(U)) \cap \mathrm{supp}(H_p\chi) \subset W.$$

Now, let $c(x,\xi)$ be a symbol of order $s+\alpha$, supported in U and elliptic at ρ_1. From Lemma 2, there exists a symbol q on U_1 of order $2s+2\alpha-1$ verifying

$$\begin{cases} H_p q = |c|^2, \\ \mathrm{supp}(q) \subset \cup_{\tau\le 0}\Phi_\tau(U). \end{cases} \tag{12}$$

Set $b = \chi q$. It is easy to verify that b is a symbol of order $2s+2\alpha-1$ satisfying

$$H_p b = |c|^2 + r,$$

where $r(x,\xi) = qH_p\chi$ is, by construction, a symbol of order $2s+2\alpha$ supported in W.

By hypothesis we have

$$\int_0^T \theta(t)(r(x,D_x)u_n, u_n)_{L^2(M)}dt \le Cte,$$

where $\theta(t) \in C_0^\infty((0,T))$.

So, to prove our result it is sufficient to show that

$$\Big|\int_0^T \theta(t)([\Delta, B]u_n, u_n)_{L^2(M)}dt\Big| \le Cte.$$

Set $P = i\partial_t - \Delta + iB_a$ and $A(t,x,D_x) = \theta(t)B(x,D_x)$, where $B(x,D_x) = Op(b)$. Denoting $(,)$ the inner product in $L^2((0,T)\times M)$, we have

$$\begin{aligned}(Pu_n, A^\star u_n) - (Au_n, Pu_n) &= ([A,\Delta]u_n, u_n) - i(\theta' Bu_n, u_n) \\ &+ 2i(B_a A u_n, u_n) + i([A, B_a]u_n, u_n).\end{aligned} \tag{13}$$

Since (u_n) and (Pu_n) are uniformly bounded in $L^2([0,T], H^s(M))$, the left hand side of (13) $(\theta' Bu_n, u_n)$ and $([A, B_a]u_n, u_n)$ are uniformly bounded. It remains to verify $|(B_a A u_n, u_n)| \le Cte$. We have

$$(B_a A u_n, u_n) = ((1-\Delta)^\alpha A a u_n, a u_n) + ((1-\Delta)^\alpha [a, A]u_n, a u_n)\,.$$

Using Lemma 1, one has

$$\begin{aligned}|((1-\Delta)^\alpha A a u_n, a u_n)| &= |((1-\Delta)^{-(s/2+\alpha/2)} A a u_n, (1-\Delta)^{(s/2+\alpha/2)} a u_n)| \\ &\le c\|a u_n\|^2_{L^2([0,T],H^{s+\alpha})} \le Cte\,.\end{aligned}$$

On the other hand $(1-\Delta)^\alpha[a,A]$ is of order $2s+2\alpha-1$ and $2\alpha - 1 \le 0$ then $((1-\Delta)^\alpha[a,A]u_n, a u_n)$ is uniformly bounded.

Consequently

$$|([\Delta, A]u_n, u_n)| = \Big|\int_0^T \theta(t)([\Delta, B]u_n, u_n)_{L^2(M)}dt\Big| \le Cte$$

which yields the desired result. $\square$

4. Proof of Theorem 1

To prove our theorem we will give the following proposition, which yields (3) and (4) and will be useful to the proof of (5).

Proposition 3

Under hypotheses of Theorem 1, consider $T > 0$, $u_0 \in H^s(M)$ $(s \in \mathbb{R})$, $f \in L^2([0,T], H^s(M))$ *and* u *solution of (8) then*

$$u \in L^2_{\text{loc}}((0,T), H^{s+\alpha}(M)). \tag{14}$$

Furthermore, for every $\theta \in C_0^\infty((0,T))$, *there exists* $c > 0$ *such that*

$$\|\theta u\|_{L^2(\mathbb{R},H^{s+\alpha}(M))} \leq c(\|u_0\|_{H^s(M))} + \|f\|_{L^2([0,T],H^s(M))}) \tag{15}$$

Proof of Proposition 3. We first prove (14). Fix $\rho_0 \in T^\star(M)$ and denote γ the bicharacteristic issued from ϱ_0. By the geometric control assumption, γ intersects the region $T^\star(\omega)$ in some point ρ_1. Therefore by Lemma 1 $u \in L^2_{\text{loc}}((0,T), H^{s+\alpha}(\rho_1))$. Applying the regularity propagation result (Proposition 2), we conclude that $u \in L^2_{\text{loc}}((0,T), H^{s+\alpha}(\rho_0))$. This is true for all $\rho_0 \in T^\star(M)$, hence $u \in L^2_{\text{loc}}((0,T), H^{s+\alpha}(M))$.

To prove the estimate (15), we consider

$$\begin{aligned}\Psi : H^s(M) \times L^2([0,T], H^s(M)) &\longrightarrow L^2(\mathbb{R}, H^{s+\alpha}(M))\\ (u_0, f) &\longmapsto \theta(t)\Big(e^{tA_a}u_0 + \int_0^t e^{(t-\tau)A_a} f(\tau,.)d\tau\Big)\end{aligned}$$

where $\theta \in C_0^\infty((0,T))$.

By the argument above, Ψ is well defined and continuous by virtu of the closed graph theorem. This completes the proof of (15). □

Now we come to the proof of (5). Let $v_0 \in H^s(M)$, using Proposition 3,

$$v = e^{tA_a}v_0 \in L^2_{\text{loc}}((0,+\infty), H^{s+\alpha}(M))$$

so

$$v(t,.) \in H^{s+\alpha}(M), \qquad \text{for almost all} \quad t \in (0,+\infty).$$

Using again the previous result, we obtain $v \in L^2_{\text{loc}}((0,+\infty), H^{s+2\alpha}(M))$. By iteration we conclude that

$$v \in L^2_{\text{loc}}((0,+\infty), H^k(M)), \qquad \forall k \in \mathbb{N}.$$

Using the equation satisfied by v, we check easily that

$$\partial_t v \in L^2_{\text{loc}}((0,+\infty), H^k(M)),$$

which means

$$v \in H^1_{\text{loc}}((0,+\infty), H^k(M)).$$

Repeating this process, we obtain

$$v \in H^k_{\text{loc}}((0,+\infty), H^k(M)), \qquad \forall k \in \mathbb{N}.$$

So we conclude that $v \in C^\infty((0,+\infty) \times \Omega)$.

Finally, the closed graph theorem applied to the map

$$\begin{array}{llll} \phi & : H^s(M) & \to & H^k(\mathbb{R} \times M) \\ & v_0 & \mapsto & \varphi(t) e^{tA_a} v_0 \end{array}$$

where $k \in \mathbb{N}$ and $\varphi \in C_0^\infty((0, +\infty))$, and the classical Sobolev embedding yield the estimate (6) and conclude the Proof of Theorem 1. $\square$

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