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ACM embeddings of curves of a quadric surface

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Abstract

Let C be a smooth integral projective curve admitting two pencils g^1_a and g^1_b such that $g^1_a + g^1_b$ is birational. We give conditions in order that the complete linear system $|sg^1_a + rg^1_b|$ be normally generated or very ample.

1. Introduction

Given a smooth projective curve C, the most natural embeddings of C in projective spaces are the projectively normal ones, i.e. those such that the restriction maps $H^0\mathscr{O}_{\mathbb{P}^n}(m) \to H^0\mathscr{O}_C(m)$ are surjective for all $m \geq 0$. It is a natural problem to construct such embeddings with $\mathbb{P}^n$ of smallest possible dimension. If $\mathscr{L}$ is a very ample line bundle on C such that the linear system $|\mathscr{L}|$ provides a projectively normal embedding one says that $\mathscr{L}$ (or the linear system $|\mathscr{L}|$) is normally generated. A famous theorem of M. Noether says that, for non-hyperelliptic curves, the canonical line bundle is normally generated. A well-known general criterion for the normal generation of a line bundle is Theorem 1 of [6] where it is proven that if $\mathscr{L}$ is very ample and $\deg \mathscr{L} \geq 2g + 1 - 2h^1\mathscr{L} - \operatorname{Cliff} C$ then $\mathscr{L}$ is normally generated. Moreover in Theorem 3 of [6] a necessary and sufficient condition for the normal generation of a line bundle $\mathscr{L}$ is given, provided that $\deg \mathscr{L}$ is greater or equal approximately $3g/2$. If one looks for normally generated (possibly special) line bundles of smaller degrees, one is forced to consider particular line bundles, whose existence depends on geometrical properties of the curve C. The most basic geometrical property of a curve

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is the existence of pencils of divisors g^1_a on C. In this paper we deal with curves C on which there exist two independent pencils g^1_a and g^1_b. By independent pencils we mean that $g^1_a + g^1_b$, i.e. the smallest linear system containing the sums $D + E$ with $D \in g^1_a$ and $E \in g^1_b$, defines a map to $\mathbb{P}^3$ birational on the image. This image will be an integral curve in a smooth quadric $\mathbf{Q} \subset \mathbb{P}^3$ of class $(a, b) \in \operatorname{Pic} \mathbf{Q} = \mathbb{Z} \oplus \mathbb{Z}$. In this case one can produce projectively normal embeddings of C by some of the complete linear systems $|sg^1_a + rg^1_b|$, i.e. the complete linear systems associated to a divisor of the form $sD + rE$ with $D \in g^1_a$ and $E \in g^1_b$, often in degrees not satisfying Green and Lazarsfeld conditions above. The main object of this paper is to give conditions for $r, s \in \mathbb{Z}$ such that the linear systems $|sg^1_a + rg^1_b|$ are very ample and normally generated. We reduce the problem to studying the linear systems $|\mathscr{O}_C(r, s)|$ on curves on $\mathbb{P}^1 \times \mathbb{P}^1$, by means of the birational embedding above. For $(r, s) \geq (0, 0)$ we are able to completely classify which $|\mathscr{O}_C(r, s)|$ are very ample and normally generated. We also make a complete study of the very ampleness of $|\mathscr{O}_C(r, -1)|$ with $r > 0$ and we give many examples of curves $C \subset \mathbf{Q}$ such that $\mathscr{O}_C(r, -1)$ is not normally generated.

The computation of the dimension of the linear systems $|\mathscr{O}_C(r, -s)|$ for $r \geq a$, $s \geq 2$ is the object of a conjecture on the multiplication structure of the deficiency module of $\mathbf{Q}$ (see 8.1). More precisely, we conjecture that such line bundles have natural cohomology if C is general and we prove it in the extremal cases (Proposition 8.2).

In Section 5 we apply our results to produce projectively normal embeddings of general trigonal curves into projective spaces $\mathbb{P}^n$ with $n \approx g/3$.

2. Preliminaries

In this paper by ***curve*** we mean a 1-dimensional locally Cohen-Macaulay projective scheme defined over an algebraically closed field k. First of all we state a very ampleness criterion for linear systems on a reduced curve.

Let $\mathscr{L}$ be a line bundle on a curve C and let $\Sigma \subseteq H^0\mathscr{L}$ be a vector sub-space of global sections. Let $Z \subset C$ be any zero-dimensional sub-scheme. Once an isomorphism $\mathscr{L} \otimes \mathscr{O}_Z \cong \mathscr{O}_Z$ is fixed, we will call the restriction map $\Sigma \otimes \mathscr{O}_C \to \mathscr{L} \otimes \mathscr{O}_Z \cong \mathscr{O}_Z$ the ***evaluation map***. When it is onto, we will say that Σ ***separates*** Z.

Lemma 2.1

Let C be any reduced curve and let $|\Sigma| \subset |H^0\mathscr{L}|$ be any linear system on C. Then $|\Sigma|$ embeds C as a closed sub-scheme of some projective space if and only if Σ separates Z for any length 2 zero-dimensional sub-scheme $Z \subset C$.

We omit the easy proof of this lemma.

A line bundle $\mathscr{L}$ on a curve C, or its associated linear system $|\mathscr{L}|$, is said to be ***normally generated*** if $\mathscr{L}$ is very ample and the curve $C' \subset \mathbb{P}(H^0\mathscr{L}) = \mathbb{P}^n$, image of C by a map associated to the complete linear system $|\mathscr{L}|$, is arithmetically Cohen-Macaulay (ACM for short). This means that for any $m \geq 1$ the multiplication map $H^0\mathscr{L} \otimes H^0\mathscr{L}^m \to H^0\mathscr{L}^{m+1}$ is surjective or, equivalently, that $H^1_*\mathscr{I}_{C'} = \oplus_{n \geq 0} H^1\mathscr{I}_{C'}(n) = 0$, with $\mathscr{I}_{C'}$ the ideal sheaf of C'. When C' is smooth the ACM property coincides with projective normality.

We recall that if $H^1\mathscr{L} = 0$ and $\mathscr{L}$ is very ample, then $\mathscr{L}$ is normally generated if $H^0\mathscr{L} \otimes H^0\mathscr{L} \to H^0\mathscr{L}^2$ is surjective; this is a consequence of the base point free pencil trick (see for example [1, Chapter III, Section 3]).

Now we will consider birational morphisms $\eta : C \to \overline{C}$ between integral curves. By means of the following lemma, one is able to transfer the property of very ampleness and normal generation of non-special line bundles on $\overline{C}$ to their pull-backs on C.

Lemma 2.2

Let $\overline{C}$ be an integral projective curve and let $\eta : C \to \overline{C}$ be a birational morphism. Let $\mathscr{L}$ be a line bundle on $\overline{C}$ with $H^1\mathscr{L} = 0$. If $\mathscr{L}$ is very ample on $\overline{C}$ then $\eta^*\mathscr{L}$ is very ample on C and if $\mathscr{L}$ is normally generated on $\overline{C}$ then $\eta^*\mathscr{L}$ is normally generated on C.

Proof. Let us consider the exact sequence of sheaves $0 \to \mathscr{O}_{\overline{C}} \to \eta_*\mathscr{O}_C \to \mathscr{N} \to 0$, with $\mathscr{N}$ a sheaf supported on the singular locus of $\overline{C}$. If $l = \text{length}(\mathscr{N})$, then $l = p_a(\overline{C}) - p_a(C)$. Moreover, from the exact sequence above we get $0 \to \mathscr{L} \to \eta_*\eta^*\mathscr{L} \to \mathscr{N} \to 0$, hence, by the hypothesis that $H^1\mathscr{L} = 0$, we get $H^1(\eta^*\mathscr{L}) = 0$ and the exact sequence

$$0 \to H^0\mathscr{L} \to H^0(\eta^*\mathscr{L}) \to H^0\mathscr{N} \to 0.$$

In particular $h^0(\eta^*\mathscr{L}) = h^0\mathscr{L} + l$ and we set $\mathbb{P}^r = \mathbb{P}(H^0\mathscr{L})$, $\mathbb{P}^{r+l} = \mathbb{P}(H^0(\eta^*\mathscr{L}))$. Since $|\mathscr{L}|$ is very ample on $\overline{C}$ it induces an embedding $\psi : \overline{C} \to \mathbb{P}^r$, and $|\eta^*\mathscr{L}|$ is base-point free because $|\mathscr{L}| \subset |\eta^*\mathscr{L}|$ is. So we get a map $\widetilde{\psi} : C \to \mathbb{P}^{r+l}$ and a projection $\pi : \mathbb{P}^{r+l} \to \mathbb{P}^r$, with vertex $\mathbb{P}(H^0\mathscr{N})$, such that $\pi \circ \widetilde{\psi} = \psi \circ \eta$. Denoting $\widetilde{C} = \operatorname{Im}\widetilde{\psi}$, we get that η decomposes as $C \to \widetilde{C} \to \overline{C}$, and from the exact sequence

$$0 \to H^0\mathscr{O}_C(1) \to H^0\mathscr{O}_{\widetilde{C}}(1) \to H^0\mathscr{N} \to 0,$$

it easily follows that $p_a(\overline{C}) - p_a(\widetilde{C}) = l$, hence $p_a(\widetilde{C}) = p_a(C)$. It follows that $\widetilde{\psi} : C \to \widetilde{C}$ is an isomorphism, i.e. $|\eta^*\mathscr{L}|$ is very ample.

Now assume $\mathscr{L}$ normally generated on $\overline{C}$. Since we know that $H^1(\eta^*\mathscr{L}) = 0$ and $\eta^*\mathscr{L}$ very ample, to show that $\eta^*\mathscr{L}$ is normally generated it is sufficient to show that the multiplication map $Sym^2H^0(\eta^*\mathscr{L}) \to H^0(\eta^*\mathscr{L}^2)$ is onto. Consider the exact sequence

$$0 \to \mathscr{M} \to H^0\mathscr{L} \otimes \mathscr{L} \to \mathscr{L}^2 \to 0,$$

with the last map given by evaluation of sections and $\mathscr{M}$ its kernel, which is a rank r vector bundle on $\overline{C}$. From the assumptions on $\mathscr{L}$ one sees that $H^1\mathscr{M} = 0$. Moreover we can consider an analogous exact sequence $0 \to \mathscr{M}' \to H^0\mathscr{L} \otimes \eta^*\mathscr{L} \to \eta^*\mathscr{L}^2 \to 0$, because $H^0\mathscr{L}$, as a subspace of $H^0(\eta^*\mathscr{L})$, generates $\eta^*\mathscr{L}$ since it has no base points. From the exact sequences

$$0 \to \mathscr{L} \to \eta_*\eta^*\mathscr{L} \to \mathscr{N} \to 0 \quad \text{and} \quad 0 \to \mathscr{L}^2 \to \eta_*\eta^*\mathscr{L}^2 \to \mathscr{N} \to 0,$$

we deduce a commutative diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & H^0\mathscr{M} \arrow[r] \arrow[d] & H^0\mathscr{L}\otimes H^0\mathscr{L} \arrow[r] \arrow[d] & H^0\mathscr{L}^2 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & H^0\mathscr{M}' \arrow[r] \arrow[d] & H^0\mathscr{L}\otimes H^0\eta^*\mathscr{L} \arrow[r] \arrow[d] & H^0\eta^*\mathscr{L}^2 \arrow[r] \arrow[d] & H^1\mathscr{M}' \\
0 \arrow[r] & H^0Q' \arrow[r] \arrow[d] & H^0\mathscr{L}\otimes H^0\mathscr{N} \arrow[r] \arrow[d] & H^0\mathscr{N} \arrow[r] \arrow[d] & H^1Q' \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Here Q' is defined as the cokernel of the inclusion $\mathscr{M}\to\eta_*\mathscr{M}'$, and it has 0-dimensional support since $\mathscr{M}$ and $\eta_*\mathscr{M}'$ have the same rank. The exactness of the third column is consequence of the fact that $H^1\mathscr{L}^2=0$. The exactness of the first column is provided by the fact that $H^1\mathscr{M}=0$; moreover we know that $H^1Q'=0$, hence also $H^1\mathscr{M}'=0$. The result then follows by looking at the second row. $\square$

3. The deficiency module

Let $\mathbf{Q}=\mathbb{P}^1\times\mathbb{P}^1$. We introduce the bigraded ring $S=\bigoplus_{a,b\in\mathbb{N}}H^0\mathscr{O}_Q(a,b)=\bigoplus S_{a,b}$ and the bi-graded S-module

$$M_Q=\bigoplus_{m,n\in\mathbb{Z}}H^1\mathscr{O}_Q(m,n)=\bigoplus_{m,n\in\mathbb{Z}}M_{Q,m,n}.$$

We collect all the information we need about M_Q in the following theorem, a simple consequence of Künneth decomposition formulas.

Theorem 3.1

The vector space structure of M_Q is given by

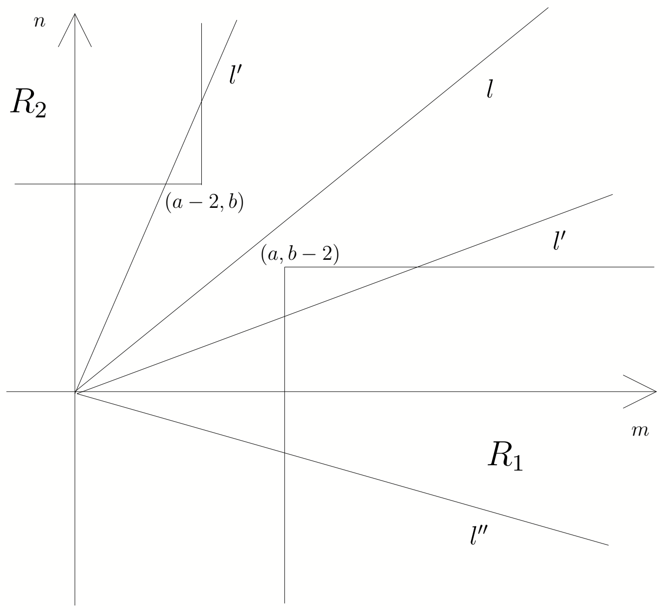
$$M_{Q,m,n}=\begin{cases} H^0\mathscr{O}_{\mathbb{P}^1}(m)\otimes H^1\mathscr{O}_{\mathbb{P}^1}(n) & \text{for } m\geq 0,\ n\leq -2\\ H^1\mathscr{O}_{\mathbb{P}^1}(m)\otimes H^0\mathscr{O}_{\mathbb{P}^1}(n) & \text{for } m\leq -2,\ n\geq 0\,.\end{cases}$$

Moreover the multiplication maps

$$\mu: M_{Q,m,n}\otimes S_{a,b}\to M_{Q,m+a,n+b}$$

are surjective.

For curves on the quadric $\mathbf{Q}$ we refer to [3] for notation, properties and further references. If $C \subset \mathbf{Q}$ is a curve of type $(a, b) \geq (1, 1)$ we call *deficiency module* of C the bigraded S-module $D_C = M_Q(-a, -b)$. Note that D_C does not vanish only when $m \geq a$ and $n \leq b - 2$, or when $m \leq a - 2$ and $n \geq b$. In the picture below the set of couples (r, s) where D_C does not vanish is the union of the two rectangles R_1, R_2. We also illustrate the various possibilities for the half-lines $l = \{(\rho r, \rho s) : \rho \in \mathbb{N}\}$ to intersect the rectangles R_1,R_2.



4. Study of $|sg_a^1 + rg_b^1|$ with $(r, s) \geq (1, 1)$

We start with the case of an integral curve $C \subset \mathbf{Q}$ of type (a, b).

Theorem 4.1

Let $C \subset \mathbf{Q}$ be a curve of type (a, b), let $(r, s) \geq (1, 1)$. Then

1) *if no multiple of (r, s) belongs to $R_1 \cup R_2$ then $\mathscr{O}_C(r, s)$ is normally generated;*
2) *if $(r, s) \notin R_1 \cup R_2$ but $\rho(r, s) \in R_1 \cup R_2$ for some $\rho > 1$ then $\mathscr{O}_C(r, s)$ is not normally generated;*
3) *if $(r, s) \in R_1 \cup R_2$, then $\mathscr{O}_C(r, s)$ is normally generated.*

Proof. First we observe that $|\mathscr{O}_Q(r, s)|$ always embeds $\mathbf{Q}$ as a projectively normal surface. This is an easy consequence of the fact that the projective coordinate ring $R_Q = \bigoplus_\rho H^0 \mathscr{O}_Q(\rho r, \rho s)$ is generated in degree $\rho = 1$. In case *1)*, denoting $\mathscr{I}_C$ the ideal sheaf of C in $\mathbf{Q}$, one has $H^1 \mathscr{I}_C(\rho r, \rho s) = 0$ for any $\rho \geq 1$, hence the restriction

$H^0\mathscr{O}_Q(\rho r, \rho s) \to H^0\mathscr{O}_C(\rho r, \rho s)$ is surjective. So the ring $R_C = \bigoplus_\rho H^0\mathscr{O}_C(\rho r, \rho s)$ is generated in $\rho = 1$ and the curve C is embedded by $|\mathscr{O}_C(r,s)|$ as an ACM curve.

In case *2)* we have that the restriction map $R_Q \to R_C$ is not surjective in those degrees ρ such that $(\rho r, \rho s) \in R_1 \cup R_2$ but it is surjective at $\rho = 1$. Hence R_C cannot be generated at $\rho = 1$, so the embedding is not ACM.

Suppose now that $(r,s) \in R_1 \cup R_2$, so that $H^1\mathscr{I}_C(r,s) \neq 0$. From the structural sequence of C, twisting by (r,s) and taking cohomology, we find $H^0\mathscr{I}_C(r,s) = H^1\mathscr{O}_C(r,s) = 0$ and the exact sequence

$$0 \to H^0\mathscr{O}_Q(r,s) \to H^0\mathscr{O}_C(r,s) \to H^1\mathscr{I}_C(r,s) \to 0,$$

hence $H^0\mathscr{O}_C(r,s) \cong H^0\mathscr{O}_Q(r,s) \oplus H^1\mathscr{I}_C(r,s)$ as vector space, with dimension $h^0\mathscr{O}_C(r,s) = d+1-g$. Again, the sheaf $\mathscr{O}_C(r,s)$ is very ample and gives an embedding $C \hookrightarrow \mathbb{P}^N$ with $N = d-g$. Let $\mathscr{I}_{C'} \subset \mathscr{O}_{\mathbb{P}^N}$ be the ideal sheaf of C'.

To prove that $\gamma : H^0\mathscr{O}_{C'}(1) \otimes H^0\mathscr{O}_{\mathbb{P}^N}(n-1) \to H^0\mathscr{O}_{C'}(n)$ is surjective it is enough to prove that $\gamma' : H^0\mathscr{O}_C(r,s) \otimes H^0\mathscr{O}_Q((n-1)r,(n-1)s) \to H^0\mathscr{O}_C(nr,ns)$ is. Since $H^0\mathscr{O}_C(r,s) \cong H^0\mathscr{O}_Q(r,s) \oplus H^1\mathscr{I}_C(r,s)$, this map is the direct sum $\gamma' = \gamma'_0 \oplus \gamma'_1$, with

$$\gamma'_0 : H^0\mathscr{O}_Q(r,s) \otimes H^0\mathscr{O}_Q((n-1)r,(n-1)s) \to H^0\mathscr{O}_Q(nr,ns)$$

and

$$\gamma'_1 : H^1\mathscr{I}_C(r,s) \otimes H^0\mathscr{O}_Q((n-1)r,(n-1)s) \to H^1\mathscr{I}_C(nr,ns)$$

which are both surjective: for γ'_0 see [5, Lemma 2.3]; for γ'_1 see Theorem 3.1. □

From the theorem above and using Lemma 2.2 we deduce the following result in the abstract curves setting.

Theorem 4.2

Let C be a smooth projective curve admitting two independent pencils of divisors g^1_a and g^1_b. Let r,s be two positive integers. Then $|sg^1_a + rg^1_b|$ is normally generated if either $(r,s) \not\leq (a-2,b-2)$ and no multiple of (r,s) belongs to $R_1 \cup R_2$ or if $(r,s) \in R_1 \cup R_2$.

Remark 4.3 The result of Theorem 4.2 in some cases gives normally generated line bundles beyond Green and Lazarsfeld criterion. Consider a birational map $\eta : C \to \overline{C} \subset \mathbf{Q}$ with $\overline{C}$ of type (a,b). Then $p_a(\overline{C}) = (a-1)(b-1)$, the geometric genus $g(C) = p_a(\overline{C}) - l$ for some integer $l \geq 0$ and $\operatorname{Cliff} C \leq \min(a,b) - 2$. If we take $a \leq b$, then $\operatorname{Cliff} C \leq a-2$ and $2g(C) + 1 - \operatorname{Cliff} C \geq 2ab - 3a - 2b + 5 - 2l$. Choosing for example $r = (a-1)/2, s = b-1$, one sees that the half line $(\rho r, \rho s)$ does not intersect $R_1 \cup R_2$, hence $H^1\mathscr{O}_{\overline{C}}(r,s) = 0$. So one can apply Theorem 4.1 and Lemma 2.2 to produce the line bundle $\mathscr{L} = \eta^*\mathscr{O}_{\overline{C}}(r,s)$ on C such that $H^1(\mathscr{L}) = 0$, moreover $\deg \mathscr{L} < 2ab - 3a - 2b + 5 - 2l$ when $ab > 4a + 3b - 10 + 4l$. This can happen of course for many choices of a,b if one assumes l sufficiently small, i.e. when the two pencils g^1_a and g^1_b produce a curve $\overline{C}$ with a small singular locus.

In the particular case that $\overline{C}$ itself is smooth we do not need to apply Lemma 2.2, so we do not need to impose $H^1\mathscr{O}_{\overline{C}}(r,s) = 0$. This enables one to consider also values of $(r,s) < (a-2,b-2)$ such that the half line $(\rho r, \rho s)$ does not intersect $R_1 \cup R_2$, providing ACM embeddings $\overline{C} \subset \mathbb{P}^n$ with n small compared to $p_a(\overline{C})$.

5. Example: general trigonal curves

As an application of the preceding results, we now produce some interesting projectively normal embeddings of general trigonal curves of genus $g \geq 5$. These embeddings cannot be achieved by the methods of [6], because we will use linear systems $|\mathscr{L}|$ with high speciality $h^1\mathscr{L}$.

Notation. We adopt the notation $\lfloor t \rfloor$ for the maximum integer less or equal to t. A trigonal curve is a curve C admitting a g_3^1. The canonical model $C \subset \mathbb{P}^{g-1}$ is contained in a rational normal scroll S, isomorphic to a Hirzebruch surface $\mathbb{P}(\mathscr{O}_{\mathbb{P}^1} \oplus \mathscr{O}_{\mathbb{P}^1}(e))$, $e \geq 0$, generated by the lines $\langle D \rangle$ for any $D \in g_3^1$. The Maroni invariant m of C can be defined as the degree in $\mathbb{P}^{g-1}$ of a curve C_0 of S such that $C_0^2 = -e$, unique in the case $e > 0$. We refer to [9] for a modern account of Maroni's theory of line bundles on trigonal curves. If C is general one knows that m is maximal, more precisely $m = \lfloor (g(C) - 2)/2 \rfloor$, see [9, Corollary 5, p. 177]. We will denote by $\mathscr{T}$ the line bundle on C associated to a g_3^1 and by $\mathscr{K}_C$ the canonical line bundle.

Proposition 5.1

A general trigonal curve of even genus $g \geq 6$ can be embedded as a projectively normal curve in $\mathbb{P}^n$, with $n = 2\lfloor (g-2)/6 \rfloor + 3$ and degree $d = g + 3 - s/2$, with $s = g - 2 - 6\lfloor (g-2)/6 \rfloor$.

Proof. Let C be a general trigonal curve with $g = g(C) \geq 6$ even, so that $m = g/2 - 1$. From [9, Corollary 1, p. 175], one knows that there exists a line bundle $\mathscr{M}$ on C of degree $b = g - m = g/2 + 1$ with $h^0\mathscr{M} = 2$, base point free and not composed with $\mathscr{T}$. Using the two pencils $|\mathscr{T}| = g_3^1$ and $|\mathscr{M}| = g_b^1$ one easily sees that C can be embedded in $\mathbb{P}^1 \times \mathbb{P}^1$ as a smooth curve of type $(3, g/2 + 1)$. So we can use line bundles of the form $\mathscr{O}_C(r, s) = \mathscr{T}^s \otimes \mathscr{M}^r$ and Theorem 4.1 to find projectively normal models of C in some projective space $\mathbb{P}^n$. The smallest value of n one can achieve with this method can be obtained by taking $(r, s) = (1, t)$ with $t > (b-2)/3$, so that the half-line $(\rho r, \rho s)$ does not intersect $R_1 \cup R_2$. More precisely, we take the minimum possible such t, that is $t = \lfloor (g-2)/6 \rfloor + 1$. Since $H^0\mathscr{O}_C(1, t) = 2(t + 1)$, we find the desired result. The value of d is obtained by a straightforward computation, using the formula $d = \deg \mathscr{O}_C(1, t) = g/2 + 1 + 3t$. □

For completeness, we state without proof a similar result for g odd. This case is more subtle, since C will only admit a birational model $\overline{C} \subset \mathbf{Q}$ with one node. One can get some good projectively normal embedding of C by a suitable linear system of the form $|\mathscr{I}_{x_0,\overline{C}}(1, t)|$ with $\mathscr{I}_{x_0,\overline{C}}$ the ideal sheaf of the node $x_0 \in \overline{C}$. We omit the details, hoping to come back to similar techniques in the future, in greater generality and with more applications.

Proposition 5.2

A general trigonal curve of odd genus $g \geq 5$ can be embedded as a projectively normal curve in $\mathbb{P}^n$ with $n = 2\lfloor (g-1)/6 \rfloor + 2$ and degree $d = g + 2 - s/2$, with $s = g - 1 - 6\lfloor (g-1)/6 \rfloor$.

Question: Do the values of n in Proposition 5.1 and 5.2 give the minimum possible dimension of a projective space in which a general trigonal curve of genus $g \geq 5$ can be embedded as a projectively normal curve?

6. Study of $|rg_b^1|$

Let $C \subset \mathbf{Q}$ be an integral curve of type (a, b). For every couple $(r, -s)$, $r \geq a$, $s \geq 0$, the linear system $|\mathscr{O}_C(r, -s)|$ has degree $d = br - as$ and dimension $\geq d - g$, where $g = (a-1)(b-1)$ is the arithmetic genus of C. In order to realize $H^0\mathscr{O}_C(r, -s)$ using bi-homogeneous forms on $\mathbf{Q}$, we prove a technical lemma which will be useful also in the next sections.

Lemma 6.1

Let $C \subset \mathbf{Q}$ be an integral curve of type (a, b), let $y \geq b-1$ and $Y = C.(\sum_{i=1}^{y+s} L'_i)$, where L'_i are $y+s$ general $(0,1)$-lines. Then, for any $r \geq a$, $s \geq 0$ such that $h^0\mathscr{O}_C(r, -s) = d + 1 - g \geq 0$, the linear system $|\mathscr{O}_C(r, -s)|$ is equal to the set of divisors on C of the form $D.C - Y$ with D belonging to the linear sub-system $|\mathscr{I}_Y(r, y)| \subset |\mathscr{O}_Q(r, y)|$ of the divisors of bi-degree (r, y) passing through Y. Moreover $|\mathscr{I}_Y(r, y)|$ has no base points out of the scheme Y.

Proof. The first assertion follows by the assumption on the dimension of the series. Set $\Sigma = H^0\mathscr{I}_Y(r, y)$. It also follows that Y imposes independent conditions to $H^0\mathscr{O}_C(r, y)$. From this it is not difficult to see that the restriction Σ_Γ of Σ to any $(1,1)$-curve Γ has degree $r + y - n$, where n is the degree of the sub-scheme $\Gamma.Y$, and dimension $r + y + 1 - n$, so that Σ has no base points out of Y, including base points infinitely near to a point of Y. □

In a similar way one can prove that $H^0\mathscr{O}_C(-r, s)$ is cut by the curves of $H^0\mathscr{O}_Q(x, s)$ passing through $X = C.(\sum_{i=1}^{x+r} L_i)$, where L_i are $x + r$ general $(1,0)$-lines.

Now we study the linear system $|\mathscr{O}_C(r, 0)|$: we have

$$0 \to H^0\mathscr{O}_Q(r, 0) \to H^0\mathscr{O}_C(r, 0) \to H^1\mathscr{I}_C(r, 0) \to 0,$$

from which we see that if $r < a$ then $H^1\mathscr{I}_C(r, 0) \cong H^1\mathscr{O}_Q(r - a, -b) = 0$ so that $H^0\mathscr{O}_C(r, 0) \cong H^0\mathscr{O}_Q(r, 0)$ does not define a very ample linear system; for $r \geq a$ we get

$$h^0\mathscr{O}_C(r, 0) = h^0\mathscr{O}_Q(r, 0) + h^1\mathscr{I}_C(r, 0) = r + 1 + (r - a + 1)(b - 1) = d + 1 - g$$

where $d = rb$ is the degree of the divisor associated to this sheaf, $g = (a-1)(b-1)$ is the arithmetic genus of C.

Theorem 6.2

Let $C \subset \mathbf{Q}$ be an integral curve of type (a, b). Then the linear system $|\mathscr{O}_C(r, 0)|$ is very ample for any $r \geq a$.

Proof. By the previous lemma, taking $y \geq b - 1$, the linear system $|\mathscr{O}_C(r, 0)|$ is cut on C by the forms in $\Sigma = H^0\mathscr{I}_Y(r, y)$, with $Y = C.(L'_1 + \cdots + L'_y)$ for y general $(0,1)$-lines

L'_i. In view of Lemma 2.1, let $Z \subset C$ be a length 2 zero dimensional sub-scheme. We can assume that Z does not intersect any of the lines L'_i, since these can be chosen in a general way.

Consider any $(1,1)$-curve Γ containing Z. The argument used at the end of the proof of Lemma 6.1 shows that the restriction Σ_Γ separates the points of Γ out of the scheme Y. So the evaluation map $\Sigma \to \mathcal{O}_Z$ is onto. □

Theorem 6.3

For any $r \geq a$ the line bundle $\mathcal{O}_C(r,0)$ is normally generated.

Proof. Denoting by $\mathcal{I}_{C'} \subset \mathcal{O}_{\mathbb{P}^N}$ the ideal sheaf of C' as a subscheme of $\mathbb{P}^N$, $N = d - g$, to show that $H^1\mathcal{I}_{C'}(n) = 0$ for any n we repeat the same argument used for $\mathcal{O}_C(r,s)$ with $(r,s) \geq (1,1)$, case *3)* of Theorem 4.1. □

Theorem 6.4

Let C be a smooth projective curve admitting two independent pencils of divisors g^1_a and g^1_b. Then, for any $r \geq a$ the linear system $|rg^1_b|$ is normally generated.

Proof. This is an immediate consequence of the preceding theorem, Lemma 2.1 and Lemma 2.2, observing that $H^1\mathcal{O}_{\overline{C}}(r,0) = 0$ for $r \geq a$. □

Remark 6.5 It is not difficult to see that the results of Theorem 6.3 and Theorem 6.4 can provide examples of normally generated line bundles beyond Green's and Lazarsfeld's sufficient condition.

7. The linear systems $|rg^1_b - g^1_a|$

As in the preceding sections, we start with studying linear systems $|\mathcal{O}_C(r,-1)|$, with $r \geq 0$, on integral (a,b)-curves on $\mathbf{Q}$. We will be able to characterize the very-ampleness of such linear systems. We will also give some counterexamples for the normal generation property.

From the structural sequence of $C \subset \mathbf{Q}$ one immediately sees that $H^0\mathcal{O}_C(r,-1) \cong H^1\mathcal{O}_Q(r-a,-1-b) \neq 0$ for $r \geq a$ and $b \geq 1$, which will be assumed henceforth. Hence we know that $h^0\mathcal{O}_C(r,-1) = (r+1-a)b$ and $H^1\mathcal{O}_C(r,-1) = 0$. By Lemma 6.1 we represent $H^0\mathcal{O}_C(r,-1)$ as the restriction to C of $\Sigma = H^0\mathcal{I}_Y(r,b) \subset H^0\mathcal{O}_Q(r,b)$, with $Y = C.(L'_1 + \cdots + L'_{b+1})$, for an arbitrary choice of $b+1$ distinct $(0,1)$-lines. In particular we recall that Y imposes independent conditions to $H^0\mathcal{O}_Q(r,b)$ and that Σ has no base points out of Y. By Lemma 2.1, the very-ampleness of Σ_C is proved if one shows that Σ separates Z, for any length 2 zero-dimensional $Z \subset C$.

Theorem 7.1

Let $C \subset \mathbf{Q}$ be an integral curve of type $(a,b) \geq (1,1)$. Then for any $r > a$ the linear system $|\mathcal{O}_C(r,-1)|$ is very ample. If $r = a$, the set of of curves C of type (a,b) on which $|\mathcal{O}_C(a,-1)|$ is not very ample is closed in $|\mathcal{O}_Q(a,b)|$ and has codimension at least $b-3$. In particular if $b \geq 4$ and C is general, then $|\mathcal{O}_C(a,-1)|$ is very ample.

Proof. Take any zero-dimensional sub-scheme $Z \subset C$ of length 2. We distinguish the cases $r > a$ or $r = a$.

Case $r > a$. Assume Z is contained in a $(1, 0)$ line L, which we may assume not passing through any point of Y by a general choice of the L'_i and by the irreducibility of C. Then we consider the restriction Σ_L of Σ to L, which gives a linear system of degree b and vector space dimension equal to $\dim \Sigma - \dim \Sigma'$ with Σ' the linear sub-space of Σ of the forms divided by L. We may identify Σ' with the sub-space of $H^0\mathscr{O}_Q(r-1, b)$ of forms vanishing on Y. Since $r - 1 \geq a$ we have that Σ'_C represents $H^0\mathscr{O}_C(r-1, -1)$ and, as a consequence, Y imposes independent conditions to $H^0\mathscr{O}_Q(r-1, b)$ as well. So $\dim \Sigma_L = h^0\mathscr{O}_Q(r, b) - h^0\mathscr{O}_Q(r-1, b) = b + 1$ and therefore Σ_L is very ample on L. So Σ separates Z as well.

Now assume Z is not contained in any $(1, 0)$-line. Take any divisor D of type $(r-1, b)$ passing through Y, not intersecting Z, which exists since the space of sections $\Sigma' \subset H^0\mathscr{O}_Q(r-1, b)$ representing $|\mathscr{O}_C(r-1, -1)|$ has no base points out of Y. Consider, for any $(1, 0)$-line L intersecting Z, the divisor $D + L$; it is clear that divisors of this form separate Z, no matter whether it is reduced or not.

Case $r = a$. If Z is on a $(1, 0)$-line then by a similar reasoning as above, one sees that Σ_L has degree b and dimension $b+1$, hence it separates Z, and so does Σ. Now let Z be not contained on any $(1, 0)$-line. Assume first that $Z = Z_{\mathrm{red}} = \{P, Q\}$. We study the locus $\mathcal{C}_1$ of curves C of type (a, b) such that $|\mathscr{O}_C(a, -1)|$ does not separate some $Z \subset C$ of this type. Given a curve $C \in \mathcal{C}_1$, we fix the $(0, 1)$-lines $L'_1, \ldots, L'_{b+1}$ in a general position with respect to C, with equations $v - \alpha_i v' = 0$. Setting $F_i = \Pi_{j \neq i}(v - \alpha_i v')$, we see that C is associated to a form F of bi-degree (a, b), which can be written as $F = G_1 F_1 + \cdots + G_{b+1} F_{b+1}$, with G_i forms of bi-degree $(a, 0)$ *uniquely determined by* F. Moreover Σ has basis $G_1 F_1, \ldots, G_{b+1} F_{b+1}$ since all these forms belong to Σ and are linearly independent. Let us assume for the moment that the two $(1, 0)$-lines L_1 and L_2 passing through P and Q have equations $u = 0$ and $u' = 0$, respectively. We also denote $\Gamma = L_1 + L_2$, and observe that $\Sigma \cong \Sigma_\Gamma \cong \Sigma_{L_1} \cong \Sigma_{L_2}$ by restriction since no non-zero form in Σ is divisible by L_1 or L_2.

We consider the split exact sequence of structural sheaves

$$0 \to \mathscr{O}_{L_1} \to \mathscr{O}_\Gamma \to \mathscr{O}_{L_2} \to 0,$$

which induces a canonical isomorphism of vector spaces

$$H^0\mathscr{O}_\Gamma(a, b) \cong H^0\mathscr{O}_{L_1}(a, b) \oplus H^0\mathscr{O}_{L_2}(a, b).$$

Writing each G_i as $G_i = u^a h_{0i} + u^{a-1}u' h_{1i} + \cdots u'^a h_{ai}$, we see that

$$\begin{aligned} \Sigma_\Gamma &= \langle (u^a h_{01} + u'^a h_{a1})F_1, \ldots, (u^a h_{0\,b+1} + u'^a h_{a\,b+1})F_{b+1} \rangle \\ \Sigma_{L_1} &= \langle u'^a h_{a1} F_1, \ldots, u'^a h_{a\,b+1} F_{b+1} \rangle \\ \Sigma_{L_2} &= \langle u^a h_{01} F_1, \ldots, u^a h_{0\,b+1} F_{b+1} \rangle\,. \end{aligned}$$

Setting $\sigma_i = (u^a h_{0i} + u'^a h_{ai})F_i$, $i = 1, \ldots, b+1$ we see that the evaluation map $\Sigma_\Gamma \to \mathscr{O}_Z$ is defined by $\sigma_i \mapsto (h_{ai}F_i(P), h_{0i}F_i(Q))$. Σ does not separate Z if and only if the image of this map has dimension 1, that is the rank of the matrix

$$\begin{pmatrix} h_{a1}F_1(P) & \dots & h_{a\,b+1}F_{b+1}(P) \\ h_{01}F_1(Q) & \dots & h_{0\,b+1}F_{b+1}(Q) \end{pmatrix}$$

is equal to 1. Moreover the condition $Z \subset C$ is equivalent to $h_{a1}F_1(P) + \cdots + h_{a\,b+1}F_{b+1}(P) = 0$, or equivalently $h_{01}F_1(Q) + \cdots + h_{0\,b+1}F_{b+1}(Q) = 0$. Calling $H^{(a,0)} = (h_{ji})$, $j = 0, \dots a$, $i = 1, \dots, b+1$, the matrix whose columns give the coefficients of G_i, $i = 1, \dots, b+1$, we can compute the number of parameters on which this matrix depends. The first row $h_{01}, \dots, h_{0\,b+1}$ depends on $b+1$ affine parameters. Fixing such a row, Q varies in a 0-dimensional set. The last row $h_{a1}, \dots, h_{a\,b+1}$ is assigned by the first row and two extra parameters: the point P and a proportionality factor. The other rows add $(a-1)(b+1)$ more parameters to the dimensional count. Finally, letting L_1 and L_2 vary adds two more parameters. We find $\dim \mathcal{C}_1 \leq b+1+2+(a-1)(b+1)+2 = a(b+1)+4$ and hence $\operatorname{codim} \mathcal{C}_1 \geq b-3$.

Now we assume Z non-reduced of length 2. We will estimate the dimension of the locus $\mathcal{C}_2$ of the curves containing such a Z not separated by Σ. The sub-scheme Z is contained in $\Gamma = 2L$, a double line of type $(2,0)$ in $\mathbf{Q}$. We assume that $L = (u')_0$ for the moment. One has the exact sequence

$$0 \to \mathscr{O}_L(-1,0) \xrightarrow{u'} \mathscr{O}_\Gamma \to \mathscr{O}_L \to 0,$$

which induces a non-canonical isomorphism $H^0\mathscr{O}_\Gamma(a,b) \cong H^0\mathscr{O}_L(a,b) \oplus H^0\mathscr{O}_L(a-1,b)$. More precisely, the following is a basis on $H^0\mathscr{O}_\Gamma(a,b)$:

$$u'u^{a-1}F_1, \dots, u'u^{a-1}F_{b+1},\ u^aF_1, \dots, u^aF_{b+1},$$

where the first $b+1$ vectors come from $H^0\mathscr{O}_L(a-1,b)$ and the remaining $b+1$ restrict to a basis of $H^0\mathscr{O}_L(a,b)$. In the present case, and in the notation introduced above, Σ_Γ has basis $u^ah_{0i}F_i + u'u^{a-1}h_{1i}F_i$, $i = 1, \dots, b+1$. Now we use the hypothesis that the evaluation map $\Sigma_\Gamma \to \mathscr{O}_Z$ is not surjective. Let the support of Z be a point $P \in L$. Then the image of $\Sigma_\Gamma \to \mathscr{O}_Z$ is generated by $h_{0i}F_i(P) + \alpha u' h_{1i}F_i(P) \in \mathscr{O}_Z \cong \mathscr{O}_P \oplus u'\mathscr{O}_P$, for a given $\alpha \neq 0$, since $Z \not\subset L$. Since $F_i(P) \neq 0$ by a general choice of the L_i', we find the condition that the two rows $(h_{01}, \dots, h_{0\,b+1})$ and $(h_{11}, \dots, h_{1\,b+1})$ of the matrix $H^{(a,0)}$ defined above must be proportional. To this we add the condition that $P \in C$, which is equivalent to $h_{01}F_1(P) + \cdots + h_{0\,b+1}F_{b+1}(P) = 0$. The count of affine parameters for $H^{(a,0)}$ gives: $b+1$ parameters for the first row, 1 extra parameter for fixing also the second row, $(a-1)(b+1)$ parameters for the remaining rows, plus 1 parameter for letting the line L vary arbitrarily. We find $\dim \mathcal{C}_2 \leq a(b+1)+2$, and hence its codimension in $H^0\mathscr{O}_Q(a,b)$ is $\geq b-1$. □

Corollary 7.2

Let C be a smooth projective curve with two independent pencils g^1_a and g^1_b. Then for any $r > a$ the linear system $|-g^1_a + rg^1_b|$ is very ample.

Proof. This is an immediate consequence of the first part of the statement of the preceding theorem and Lemma 2.2. □

Note that $\deg|-g^1_a + rg^1_b| = rb - a$ and this number can be small with respect to $2g(C)$ for many $r > a$, provided that $\overline{C}$ has a sufficiently small singular locus. So the statement above is not a trivial consequence of the general very-ampleness criterion for curves.

Remark 7.3 Since imposing to a curve of type (a, b) to have one ordinary node amounts to imposing just one condition, we get from Theorem 7.1 the result that the general curve C of type (a, b) with at most $b-4$ nodes can be embedded into some projective space by $|\mathscr{O}_C(a, -1)|$.

Remark 7.4 For $\mathscr{L} = \mathscr{O}_C(r, -1)$ and C a general curve of type $(a, b) \geq (2, 2)$ on $\mathbf{Q}$, we know that $h^1\mathscr{L} = 0$, and one expects $\operatorname{Cliff} C = \min(a, b) - 2$, so, by Green-Lazarsfeld's criterion, we expect $\mathscr{L}$ to be normally generated if $rb - a \geq 2ab - 2a - 2b + 5 - \min(a, b)$. Certainly, if $\deg \mathscr{L} \geq 2g_C + 1$ then it is a classical result that $\mathscr{L}$ is normally generated. For example if $r \geq 2a$ then $\mathscr{O}_C(r, -1)$ is always normally generated.

On the other hand we can give examples of very ample line bundles $\mathscr{O}_C(r, -1)$ which are not normally generated.

Example 7.5 Let $C \subset \mathbf{Q}$ be a hyperelliptic curve of genus $g \geq 3$, of type $(2, g+1)$. The linear system $|\mathscr{O}_C(2, -1)|$ is very ample and has degree $2g$. Its embedding $C \hookrightarrow \mathbb{P}^g$ cannot be ACM by [8, Theorem 4.1 or Corollary 3.4.] This means that the analogue of *3)* in Theorem 4.1 does not hold in general.

Example 7.6 Let $C \subset \mathbf{Q}$ be an (a, b)-curve, $a, b \geq 2$, and let $C' \hookrightarrow \mathbb{P}^N$ be its image in the embedding given by $|\mathscr{O}_C(r, -1)|$ with $r \geq a$. Setting $r = a + y$ and $\mathscr{L} = |\mathscr{O}_C(r, -1)|$ we have $h^0\mathscr{L} = b(y+1) = N + 1$. From the structural sequence of C' in $\mathbb{P}^N$ we see that in some cases the map $H^0\mathscr{O}_{\mathbb{P}^N}(2) \longrightarrow H^0\mathscr{O}_{C'}(2)$ cannot be surjective because $h^0\mathscr{O}_{\mathbb{P}^N}(2) < h^0\mathscr{O}_{C'}(2)$. We have:

$$0 \to H^0\mathscr{L}^2 \to H^1\mathscr{O}_Q(2r-a, -2-b) \to H^1\mathscr{O}_Q(2r, -2) \to H^1\mathscr{L}^2 \to \dots$$

hence

$$h^0\mathscr{O}_{C'}(2) = h^0\mathscr{L}^2 \geq (2r-a+1)(b+1) - (2r+1) = ab + 2by - a + b.$$

Now, $h^0\mathscr{O}_{\mathbb{P}^N}(2) < h^0\mathscr{O}_{C'}(2)$ when

$$a > \frac{b^2(y+1)^2 - b(y+1) - 2by}{2(b-1)}.$$

In the case $r = a$ we find that if $a > b/2$, then $|\mathscr{O}_C(a, -1)|$ is not normally generated.

Indeed we can prove the following general result, in the case $r = a$.

Theorem 7.7

Let C be any curve of type (a, b) in $\mathbf{Q}$ with $b \geq 4$ and such that $|\mathscr{O}_C(a, -1)|$ is very ample. Then $\mathscr{O}_C(a, -1)$ is not normally generated.

Proof. By the result in Example 7.6, we are left with the case $b \geq 2a \geq 4$. By semi-continuity of $h^1\mathscr{I}_{C'}(2)$, it is sufficient to show that for C general of type (a, b) one has $h^1\mathscr{I}_{C'}(2) \neq 0$. Let $\phi : C \to \mathbb{P}^{b-1}$ be the embedding given by $|\mathscr{O}_C(a, -1)|$. Since C admits the pencil of divisors g^1_a given as the restriction to C of $|\mathscr{O}_Q(0, 1)|$, by [2, Theorem 2, p. 4], we know that $C' = \phi(C)$ is contained in the rational normal scroll $S = \bigcup\langle\phi(C.L')\rangle$, with L' varying among the $(0, 1)$-lines of $\mathbf{Q}$. The scroll S is the image in $\mathbb{P}^{b-1}$ of a projectivized vector bundle $\tilde{S} = \mathbb{P}(F) \xrightarrow{\pi} \mathbb{P}^1$, with $F \cong \mathscr{O}_{\mathbb{P}^1}(\beta_1) \oplus \cdots \oplus \mathscr{O}_{\mathbb{P}^1}(\beta_t)$, by the map associated to the complete linear system $|\mathscr{O}_{\tilde{S}}(1)|$ such that $\pi_*\mathscr{O}_{\tilde{S}}(1) = F$. One has $\beta_i \geq 0$ for any $i = 1, \ldots, t$, and $t \leq a$, since $\dim\langle\phi(C.L')\rangle \leq a-1$. The reader can find useful references for this discussion in [2] pages 5-8. We show that indeed $t = a$, equivalently that $\dim\langle\phi(C.L')\rangle = a - 1$. This is equivalent to show that $h^0\mathscr{O}_C(a, -1) \otimes \mathscr{O}_C(-C.L') = h^0\mathscr{O}_C(a, -1) - a$. Since $\mathscr{O}_C(a, -1) \otimes \mathscr{O}_C(-C.L') = \mathscr{O}_C(a, -2)$ and $\deg \mathscr{O}_C(a, -2) + 1 - g_C = b - a > 0$, by Proposition 8.2, proved in the next section with independent arguments, we know that $\mathscr{O}_C(a, -2)$ is non-special and the result follows.

We now evaluate $h^0\mathscr{I}_{C'}(2) \geq h^0\mathscr{I}_S(2) \geq \binom{b+1}{2} - h^0\mathscr{O}_S(2)$. We have $h^0\mathscr{O}_S(1) = h^0(F) = \deg F + a = b$ and

$$h^0\mathscr{O}_S(2) = h^0 Sym^{(2)} F = (a+1)\deg F + a(a+1)/2 = (a+1)(b-a) + a(a+1)/2.$$

Then we find that the rank of the restriction map $\rho : H^0\mathscr{O}_{\mathbb{P}^{b-1}}(2) \to H^0\mathscr{O}_{C'}(2)$ is at most $h^0\mathscr{O}_S(2) = (a+1)b - a(a+1)/2$. We also know that $h^0\mathscr{O}_{C'}(2) = h^0\mathscr{O}_C(2a, -2) = ab - a + b = (a+1)b - a$, and since $-a(a+1)/2 < -a$ for $a \geq 2$, we get that ρ cannot be surjective. $\square$

8. The linear systems $|\mathscr{O}_C(r, -s)|$ with $s \geq 2$. Open problems

The study of $|\mathscr{O}_C(r, -s)|$, with $r \geq a, s > 1$ presents some extra difficulties and is still incomplete, even for a general curve of given numerical type. The geometrical representation of the linear system $|\mathscr{O}_C(r, -s)|$ provided by Lemma 6.1 is available only if $\mathscr{O}_C(r, -s)$ is non-special. The non-speciality of $\mathscr{O}_C(r, -s)$ for C general of type (a, b) is a consequence of the following conjecture. Let us recall the exact sequence, for $r \geq a$ and $s \geq 0$:

$$0 \to H^0\mathscr{O}_C(r, -s) \to H^1\mathscr{O}_Q(r-a, -s-b) \xrightarrow{\mu_F} H^1\mathscr{O}_Q(r, -s) \to H^1\mathscr{O}_C(r, -s) \to 0 \quad (8.1)$$

with the map μ_F defined as the product with F, a form representing C.

Conjecture 8.1 For a general curve $C \subset \mathbf{Q}$ of type $(a, b) \geq (1, 1)$ the map μ_F has maximal rank.

A simple computation shows that, if $r \geq a$ and $s \geq 2$ then

$$h^1\mathscr{O}_Q(r-a, -s-b) - h^1\mathscr{O}_Q(r, -s) = \deg \mathscr{O}_C(r, -s) + 1 - p_a(C),$$

hence the conjecture implies in particular that if $\deg \mathscr{O}_C(r, -s) + 1 - p_a(C) \geq 0$, then it follows $h^1\mathscr{O}_C(r, -s) = 0$, and so also that $h^0\mathscr{O}_C(r, -s) = \deg \mathscr{O}_C(r, -s) + 1 - p_a(C)$.

It is possible to determine the matrix T associated to the map μ_F, with respect to suitable bases (see the proof of Theorem 5.1 Chapter III of [7]). In the vector space $H^1\mathscr{O}_Q(r-a,-s-b) \cong H^0\mathscr{O}_{\mathbb{P}^1}(r-a) \otimes H^1\mathscr{O}_{\mathbb{P}^1}(-s-b)$ we choose the basis

$$(u^{r-a}, u^{r-a-1}u', \dots, u'^{r-a}) \otimes (1/v^{s+b-1}v', 1/v^{s+b-2}v'^2, \dots, 1/vv'^{s+b-1}).$$

Similarly in the vector space $H^1\mathscr{O}_Q(r,-s) \cong H^0\mathscr{O}_{\mathbb{P}^1}(r) \otimes H^1\mathscr{O}_{\mathbb{P}^1}(-s)$ we choose the basis

$$(u^r, u^{r-1}u', \dots, u'^r) \otimes (1/v^{s-1}v', 1/v^{s-2}v'^2, \dots, 1/vv'^{s-1}).$$

Let H^0, $H^1, \dots, H^a$ be the rows of the matrix $H \in k^{a+1,b+1}$ such that $F = \boldsymbol{u}^a H\, {}^t\boldsymbol{v}^b$ be the bi-graded polynomial defining the curve C. Then T is the Toeplitz block matrix with $r+1$ block rows and $r-a+1$ block columns

$$T = \begin{pmatrix} K_0 & \Omega & \dots & \Omega \\ K_1 & K_0 & \dots & \Omega \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & K_0 \\ \dots & \dots & \dots\dots & \\ K_a & K_{a-1} & \dots & \dots \\ \Omega & K_a & \dots & \dots \\ \dots & \dots & \dots & \dots \\ \Omega & \Omega & \dots & K_a \end{pmatrix} \in k^{(r+1)(s-1),(r-a+1)(s+b-1)}$$

where $\Omega \in k^{s-1,s+b-1}$ is the zero matrix and each block K_i is a Toeplitz matrix depending on the i-th row $H^i = (h_{i0}, h_{i1}, \dots, h_{ib})$ of H:

$$K_i = \begin{pmatrix} h_{i0} & h_{i1} & \dots & h_{ib} & 0 & \dots & 0 \\ 0 & h_{i0} & \dots & h_{ib-1} & h_{ib} & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & h_{i0} & h_{i1} & \dots & h_{ib} \end{pmatrix} \in k^{s-1,s+b-1} \qquad i = 0, \dots, a$$

Conjecture 8.1 is of course equivalent to claim that the matrix T has maximal rank if C is general. In this perspective the following special case of Conjecture 8.1 is known.

Proposition 8.2

Conjecture 8.1 is true for $s = 2$, $r \geq a$, and for $r = a$, $s \geq 2$.

Proof. In case $s = 2$ each block of the matrix T is just a row of H, $K_i = H^i$ $(i = 0, \dots, a)$ and the rank of such Toeplitz matrices has been determined in [4]. The conclusion is that T has maximal rank if and only if the vector space $R \subset k^{a+1}$ of relations among the rows of H does not contain any element of type $(\alpha, \beta)^a$ and does not contain any Hankel i-plane. This happens for the general matrix H of the form above.

As to the second part, let C' be the curve of type (b, a) obtained from C by the automorphism of $\mathbf{Q}$ which exchanges the two rulings. By Serre duality on C we see that the sequence (8.1), for $r = a$ and $s \geq 2$, becomes (set $s - 2 = s'$)

$$0 \to H^0\mathscr{O}_{C'}(s'+b,-2) \to H^1\mathscr{O}_Q(s',-a-2) \xrightarrow{\mu_{F'}} H^1\mathscr{O}_Q(s'+b,-2)$$
$$\to H^1\mathscr{O}_{C'}(s'+b,-2) \to 0$$

hence the result follows from the first case. $\square$

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