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Orbifold principal bundles on an elliptic fibration and parabolic principal bundles on a Riemann surface, II

Indranil Biswas

School of Mathematics, Tata Institute of Fundamental Research
Homi Bhabha Road, Mumbai 400005, India
E-mail: indranil@math.tifr.res.in

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Abstract

In [6], orbifold G–bundles on a certain class of elliptic fibrations over a smooth complex projective curve X were related to parabolic G–bundles over X. In this continuation of [6] we define and investigate holomorphic connections on an orbifold G–bundle over an elliptic fibration.

1. Introduction

Let X be a connected smooth projective curve defined over the field $\mathbb{C}$ of complex numbers. Fix a finite subset $\{p_1, \cdots, p_h\} \subset X$, and to each point p_i in this subset assign a positive integer m_i. Fix an elliptic curve $C := \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau)$. Given these, in [6] we constructed an elliptic fibration

$$f : Z \longrightarrow X$$

such that C acts on Z with X as the quotient. This action is free on the complement $f^{-1}(\{p_1, \cdots, p_h\})^c$ and for each point $z \in f^{-1}(p_i)$ the isotropy subgroup is $\mathbb{Z}/m_i\mathbb{Z} \subset C$. In other words, $f^{-1}(p_i)_{\text{red}} = C/(\mathbb{Z}/m_i\mathbb{Z})$. Here $\mathbb{Z}/m_i\mathbb{Z}$ is realized as a subgroup of C by sending any $n \in \mathbb{Z}$ to n/m_i.

Let G be any connected reductive linear algebraic group defined over $\mathbb{C}$. Following [6], we call a principal G–bundle E_G over Z equipped with a lift of the action of C

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to the total space of E_G, such that the action of C on E_G commutes with the action of G on E_G, to be an orbifold G–bundle. It was shown in [6] that there is a natural bijective correspondence between orbifold G–bundles over Z and parabolic G–bundles over X with $\{p_1, \cdots, p_h\}$ as the parabolic divisor.

Let E_G be an orbifold G–bundle over Z. We will call a holomorphic connection ∇ on E_G to be an orbifold connection if the action of C on E_G preserves the connection form ∇ and the orbits in E_G, for the action C on E_G, are the horizontal lifts of the orbits in Z for the action C on Z (the details are in Section 3). We show that the Einstein–Hermitian connection on a polystable orbifold G–bundle is a flat orbifold connection provided the group G is simple (Lemma 4.2).

Take any proper parabolic subgroup $P \subset G$, where G is a simple group. Take any polystable orbifold G–bundle E_G over Z. Let ∇ be the flat orbifold connection on the orbifold G–bundle E_G given by the (unique) Einstein–Hermitian connection on it. In Section 4 we investigate the connection on the associated fiber bundle E_G/P induced by ∇ and give applications to the geometric properties of some naturally occurring line bundles over E_G/P (see Theorem 4.3 and Proposition 4.4).

In [4], parabolic principal bundles over a curve were identified with what were called in [4] as ramified principal bundles. In [7], connections on ramified principal bundles were investigated.

2. Preliminaries

We begin by recalling some notation from [6]. Let $\Lambda := \mathbb{Z} + \mathbb{Z}\tau \subset \mathbb{C}$ be a lattice; the imaginary part of $\tau \in \mathbb{C}$ is nonzero. The elliptic curve $\mathbb{C}/\Lambda$ defined over $\mathbb{C}$ will be denoted by C. Let X be a compact connected Riemann surface. Fix

$$S := \{p_1, \cdots, p_h\} \subset X$$

h distinct points of X. To each point $p_i \in S$, $i \in [1, h]$, we assign a positive integer m_i.

Given these datum, there is an elliptic fibration

$$f : Z \longrightarrow X \tag{1}$$

which is smooth over the complement $X \setminus S$ (see [6, Section 2]). The elliptic curve C acts on Z and the action is transitive on any reduced fiber of f. Furthermore, for any point $x \in X \setminus S$ the action of C on $f^{-1}(x)$ is free, and for any point $p_i \in S$, the isotropy subgroup for any point of the reduced fiber $f^{-1}(x)_{\text{red}}$ is $\mathbb{Z}/m_i\mathbb{Z} \subset \mathbb{C}/\Lambda = C$ (the details are in [6, Section 2]). Let

$$\phi : C \longrightarrow \text{Aut}(Z) \tag{2}$$

be the homomorphism defined by the action of C on Z. Here $\mathbb{Z}/m_i\mathbb{Z}$ is realized as a subgroup of $\mathbb{C}/\Lambda$ by sending any $n \in \mathbb{Z}$ to n/m_i.

Let E be an algebraic vector bundle over the surface Z in (1). By a *lift of the action of C on Z to E as vector bundle automorphisms* we mean an algebraic action of C on the total space of E

$$\widehat{\phi} : C \times E \longrightarrow E \tag{3}$$

such that for any point $t \in C$, the automorphism of the total space of E that sends any $v \in E$ to $\widehat{\phi}(t, v)$ is an isomorphism of the vector bundle E with the pulled back bundle $\phi(-t)^*E$, where ϕ is the homomorphism in (2).

An algebraic vector bundle E over Z equipped with a lift of the action of C on Z to E as vector bundle automorphisms is called an *orbifold vector bundle over* Z (see [6, Definition 2.1]).

Consider the parabolic vector bundles over X with S as the parabolic divisor and such that all the parabolic weights at any point $p_i \in S$ are integral multiples of $1/m_i$, where $\{m_i\}$ are the fixed integers associated to the points of S. There is a natural bijective correspondence between such parabolic vector bundles over X and the orbifold vector bundles over Z [6, Theorem 4.4].

Let G be a connected reductive linear algebraic group defined over the field $\mathbb{C}$ of complex numbers. An *orbifold G–bundle* over Z is a holomorphic principal G–bundle E_G over Z equipped with a lift of the action of C on Z to E_G. This means that C acts holomorphically on E_G satisfying the following two conditions:

(1) the projection of E_G to Z commutes with the actions of C on E_G and Z, and

(2) the actions of G and C on E_G commute.

In [3], a parabolic G–bundle over X with S as the parabolic divisor was defined as a functor, satisfying certain conditions, from the category of finite dimensional complex left G–modules to the category of parabolic vector bundles over X with S as the parabolic divisor (see [3, Definition 2.5]). We recall that a more general version of [6, Theorem 4.4] says that there is a bijective correspondence between the following two collections:

(1) All orbifold G–bundles E_G over Z.

(2) All parabolic G–bundles $\mathcal{F}$ on X with S as the parabolic divisor such that for any left G–module V, the parabolic vector bundle $\mathcal{F}(E_*)(V)$ with parabolic structure over S has the property that for each point $p_i \in S$, all the parabolic weights of $\mathcal{F}(E_*)(V)$ at p_i are integral multiples of $1/m_i$.

(See [6, Theorem 5.1] for the details.)

In [4] it was shown that a parabolic G–bundle is same as a ramified G–bundle. We will briefly recall the definition of a ramified G–bundle.

A *ramified G–bundle over X ramified over S* is a triple (Q, ψ, f), where

(i) Q is a smooth complex variety of complex dimension $1 + \dim_{\mathbb{C}} G$,

(ii) $\psi : Q \longrightarrow X$ is a surjective algebraic map, and

(iii) $f : Q \times G \longrightarrow Q$ is an algebraic action of the algebraic group G on the variety Q such that

(1) the composition $\psi \circ f$ coincides with $\psi \circ q_1$, where $q_1 : Q \times G \longrightarrow Q$ is the projection to the first factor,

(2) the projection ψ is smooth over $\psi^{-1}(X \setminus S)$,

(3) for each point $x \in X$, the action of G on the reduced fiber $\psi^{-1}(x)_{\rm red}$ is transitive, and for each point $x \in X \setminus S$, the action of G on $\psi^{-1}(x)$ is free (the second condition implies that $\psi^{-1}(x) = \psi^{-1}(x)_{\rm red}$ for any $x \in X \setminus S$), and

(4) for each closed point $z \in \psi^{-1}(S)$, the isotropy subgroup $G_z \subset G$ at z, for the action of G on Q, is a finite cyclic group, and furthermore, the induced action of the cyclic group G_z on the quotient line $T_zQ/T_z\psi^{-1}(S)_{\rm red}$ is faithful.

Let $\mathcal{F}(E_*)$ be a functor from the category of left G–modules to the category of parabolic vector bundles defining a parabolic G–bundle over X (see [3] for the details). Let E'_G be the ramified G–bundle over X, ramified over S, associated to $\mathcal{F}(E_*)$ by the bijective correspondence constructed in [4, Theorem 3.7] between parabolic G–bundles and ramified G–bundles. The following two conditions are equivalent:

Condition A *For any left G–module V, the parabolic vector bundle $\mathcal{F}(E_*)(V)$ with parabolic structure over S has the property that for each point $p_i \in S$, all the parabolic weights of $\mathcal{F}(E_*)(V)$ at p_i are integral multiples of $1/m_i$.*

Condition B *The ramified G–bundle E'_G over X, ramified over S, has the property that for each point $z \in \psi^{-1}(p_i)_{\rm red}$, where $p_i \in S$, the order of the isotropy subgroup $G_z \subset G$ is a submultiple of m_i.*

That the above two conditions are equivalent is an immediate consequence of the construction of the bijective correspondence done in [4, Theorem 3.7].

So, on the one hand from [4, Theorem 3.7] we have a bijective correspondence between the parabolic G–bundles over X satisfying the above Condition A and the ramified G–bundles over X satisfying the above Condition B. On the other hand, from [6] we have a bijective correspondence between the orbifold G–bundles over Z and the parabolic G–bundles over X satisfying Condition A. Combining these two we get a bijective correspondence between the orbifold G–bundles over Z and the ramified G–bundles over X satisfying Condition B. This resulting bijective correspondence is easy to describe, as shown in the following theorem.

Theorem 2.1

Given an orbifold G–bundle E_G over Z, the corresponding ramified G–bundle over X is the quotient space E_G/C for the action of the elliptic curve C on E_G. Conversely, given a ramified G–bundle E'_G over X satisfying the above Condition B, there is a unique orbifold G–bundle E_G over Z such that $E'_G = E_G/C$.

If E_G is an orbifold G–bundle over Z then it is easy to check that the quotient space E_G/C is a ramified G–bundle that satisfies the Condition B. The converse is a straight–forward consequence of the constructions of the two bijective correspondences in [6] and [4].

3. Connection on orbifold bundles

Let G be a connected reductive linear algebraic group defined over $\mathbb{C}$. Let $\mathfrak{g}$ denote the Lie algebra of G. Let E_G be an orbifold G–bundle over the surface Z in (1). We recall that a usual *connection* on E_G is a $\mathfrak{g}$–valued smooth $(1,0)$–form on the total space of E_G such that

(1) the $\mathfrak{g}$–valued differential form is G–equivariant with G acting on $\mathfrak{g}$ as conjugations, and
(2) for each point $z \in Z$, the restriction of the differential form to the fiber $(E_G)_z$ coincides with the Maurer–Cartan form on $(E_G)_z$ (see the remark below).

The connection is called *holomorphic* if the $(1,0)$–form defining the connection is holomorphic.

Remark 3.1 If a Lie group H acts transitively on a manifold M such that for any point $y \in M$ the isotropy group is a finite subgroup of H, then the action of H on M identifies the Lie algebra $\mathfrak{h}$ of H with the tangent space T_yM for each point $y \in M$. This identification of $\mathfrak{h}$ with T_yM, $y \in M$, defines a $\mathfrak{h}$–valued smooth one–form on M. To see this first recall that a $\mathfrak{h}$–valued one–form on M gives a homomorphism from T_yM to $\mathfrak{h}$ for all $y \in M$. The $\mathfrak{h}$–valued smooth one–form on M defined by the action of H on M sends T_yM to $\mathfrak{h}$ using the above identification of T_yM with $\mathfrak{h}$. This $\mathfrak{h}$–valued differential form on M is called the *Maurer–Cartan form*.

Let $\mathrm{ad}(E_G)$ be the adjoint bundle of E_G. So $\mathrm{ad}(E_G)$ is the quotient $(E_G \times \mathfrak{g})/G$, where the action of G is defined as follows: the action of any $\alpha \in G$ sends any point $(z,v) \in E_G \times \mathfrak{g}$ to $(z\alpha, \mathrm{Ad}(\alpha^{-1})v)$. The adjoint bundle $\mathrm{ad}(E_G)$ is a holomorphic vector bundle over Z with its fibers having the structure of a Lie algebra isomorphic to $\mathfrak{g}$. Let

$$0 \longrightarrow \mathrm{ad}(E_G) \longrightarrow \mathrm{At}(E_G) \xrightarrow{\gamma} TZ \longrightarrow 0 \tag{4}$$

be the *Atiyah exact sequence* over Z, where TZ is the holomorphic tangent bundle of Z. We recall that for any analytic open subset $U \subset Z$, the space of all holomorphic sections of the Atiyah bundle $\mathrm{At}(E_G)$ over U is the space of all G–invariant holomorphic vector fields on $E_G|_U$; the homomorphism γ is obtained from the differential of the projection of E_G to Z (see [2] for the details). A connection on E_G is same as a C^∞ splitting of the exact sequence (4) of holomorphic vector bundles. A C^∞ splitting of (4) is a C^∞ homomorphism of vector bundles

$$\delta : TZ \longrightarrow \mathrm{At}(E_G) \tag{5}$$

such that $\gamma \circ \delta = \mathrm{Id}_{TZ}$, where γ is as in (4)

If δ in (5) is a holomorphic homomorphism, then the splitting δ is said to be holomorphic. A splitting of (4) gives a holomorphic connection on E_G if and only if the splitting is holomorphic [2].

The action of the elliptic curve C on the orbifold G–bundle E_G gives a holomorphic line subbundle

$$\mathcal{L} \subset TE_G \tag{6}$$

of the holomorphic tangent bundle of E_G. At each point $z \in E_G$, the fiber $\mathcal{L}_z \subset T_z E_G$ is the tangent space to the orbit, for the action of C on E_G, passing through z. Note that the line bundle $\mathcal{L}$ is canonically identified with the trivial line bundle over E_G with fiber $T_0 C = \mathbb{C}$ (the isotropy subgroup at z for the action of C on E_G is a finite group and hence $T_0 C$ gives a line in $T_z E_G$).

DEFINITION 3.2 An *orbifold connection* on E_G is a usual connection ∇ on the G–bundle E_G such that

(1) the action of the elliptic curve C on E_G preserves the $\mathfrak{g}$–valued differential form ∇ on E_G, and
(2) $\nabla(\mathcal{L}) = 0$, where $\mathcal{L}$ is the line subbundle defined in (6).

From the construction of the Atiyah bundle $\mathrm{At}(E_G)$ it follows immediately that the line subbundle $\mathcal{L}$ in (6) gives a holomorphic line subbundle

$$\mathcal{L}' \subset \mathrm{At}(E_G) \tag{7}$$

($\mathcal{L}'$ is generated by the sheaf of G–invariant vector fields on E_G that lie in $\mathcal{L}$). The line bundle $\mathcal{L}'$ is canonically identified with the trivial line bundle over Z with fiber $T_0 C = \mathbb{C}$.

Exactly as shown in (6), the action of C on Z gives a holomorphic line subbundle

$$\mathcal{L}'' \subset TZ \tag{8}$$

of the holomorphic tangent bundle of Z. At each point $z \in Z$, the fiber $\mathcal{L}''_z \subset T_z Z$ is the tangent space to the orbit, for the action of C on Z, passing through z. Just like $\mathcal{L}$, the line bundle $\mathcal{L}''$ is canonically identified with the trivial line bundle over Z with fiber $T_0 C = \mathbb{C}$. So we have a commutative diagram

$$\begin{array}{ccccccccc} & & & & 0 & & 0 & & \\ & & & & \downarrow & & \downarrow & & \\ & & & & \mathcal{L}' & = & \mathcal{L}'' & & \\ & & & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathrm{ad}(E_G) & \longrightarrow & \mathrm{At}(E_G) & \stackrel{\gamma}{\longrightarrow} & TZ & \longrightarrow & 0 \end{array} \tag{9}$$

obtained from (4), where $\mathcal{L}'$ and $\mathcal{L}''$ are constructed in (7) and (8) respectively; the isomorphism in (9) of $\mathcal{L}'$ with $\mathcal{L}''$ is obtained from the fact that both the line bundles are identified with the trivial line bundle over Z with fiber $T_0 C = \mathbb{C}$.

Note that the action of C on E_G induces an action of C on the exact sequence (4); each vector bundle in (4) is an orbifold vector bundle and all the homomorphisms commute with the actions of C.

The proof of the following proposition is straight–forward.

Proposition 3.3

A connection on E_G defined by a splitting δ (as in (5)) of the Atiyah exact sequence (4) gives an orbifold connection if and only if the following two conditions hold:

(1) *the homomorphism δ commutes with the actions of C on TZ and $\mathrm{At}(E_G)$,*

(2) *the restriction of the homomorphism δ to $\mathcal{L}'' \subset TZ$ is the identity map $\mathcal{L}' = \mathcal{L}''$ in (9).*

The first (respectively, second) condition in the above proposition is equivalent to the first (respectively, second) condition in Definition 3.2.

A *holomorphic orbifold connection* on E_G is an orbifold connection on E_G such that the underlying connection is holomorphic.

Lemma 3.4

The curvature of a holomorphic orbifold connection defined on an orbifold G–bundle E_G over Z vanishes identically.

Proof. Let E_G be an orbifold G–bundle E_G over Z and ∇ a holomorphic connection on E_G. The curvature $K(\nabla) = \nabla^2 = \nabla \circ \nabla$ is a holomorphic section of $\Omega^2_Z \otimes \mathrm{ad}(E_G)$.

Now, if ∇ is an orbifold connection, then for any point $z \in Z$ and any tangent vector $v \in T_z Z$ in the orbit, passing through z for the action of C on Z, we have

$$i_v K(\nabla)_z = 0\,,$$

where $i_v K(\nabla)_z \in (\Omega^1_Z \otimes \mathrm{ad}(E_G))_z$ is the contraction of the two–form $K(\nabla)_z$ by the tangent vector v. Since the complex dimension of Z is two, this immediately implies that $K(\nabla)_z = 0$. This completes the proof of the Lemma. □

The following proposition says that an orbifold connection on E_G descends to a $\mathfrak{g}$–valued one–form on the quotient $E'_G := E_G/C$ which is a ramified G–bundle over X (see Theorem 2.1).

Proposition 3.5

Let ∇ be a $\mathfrak{g}$–valued holomorphic one–form on E_G defining a holomorphic orbifold connection on an orbifold G–bundle E_G over Z. Then there is unique $\mathfrak{g}$–valued holomorphic one–form ∇' on $E'_G := E_G/C$ such that $q^*\nabla' = \nabla$, where $q : E_G \longrightarrow E'_G$ is the quotient map. This holomorphic form ∇' on E'_G satisfies the following two conditions:

(1) *the $\mathfrak{g}$–valued differential form ∇' is G–equivariant with G acting on $\mathfrak{g}$ as conjugations (the action of G on E_G induces an action of G on E'_G), and*

(2) *for any point $x \in X$, the restriction of ∇' to the reduced fiber of E'_G over x coincides with the Maurer–Cartan form on the fiber (see Remark 3.1).*

Conversely, let θ' be any $\mathfrak{g}$–valued holomorphic one–form on $E'_G := E_G/C$ satisfying the above two conditions. Then the $\mathfrak{g}$–valued holomorphic one–form $q^*\theta'$ on E_G defines a holomorphic orbifold connection on E_G.

Proof. Let ∇ be a $\mathfrak{g}$–valued holomorphic one–form on the total space of E_G defining a holomorphic orbifold connection on E_G. Since the differential form ∇ is C–invariant and it vanishes along the orbits for the action of C on E_G, it is straight–forward to deduce that there is a smooth $\mathfrak{g}$–valued $(1\,,0)$–form ∇' on $E'_G := E_G/C$ such that $q^*\nabla' = \nabla$, where q is the projection of E_G to E'_G. Note that for the map $f : \mathbb{D} \longrightarrow \mathbb{D}$ defined by $z \longmapsto z^n$, where $\mathbb{D} \subset \mathbb{C}$ is the unit disk, any holomorphic one–form on $\mathbb{D}$ invariant under the action of the Galois group $\mathbb{Z}/n\mathbb{Z} = \mathrm{Gal}(f)$ must be of the form $g(z)z^{n-1}\mathrm{d}z$, where $g(z)$ is a holomorphic function on $\mathbb{D}$ invariant under the action of the Galois group $\mathrm{Gal}(f) = \mathbb{Z}/n\mathbb{Z}$. Since invariant holomorphic forms descends for a quotient by a finite group, there is a $\mathfrak{g}$–valued $(1\,,0)$–form ∇' on E'_G with $q^*\nabla' = \nabla$. See [5, page 525, Lemma 4.11] for the details of the construction.

As the differential form ∇ is holomorphic, the form ∇' is also holomorphic. Since the form ∇ is G–equivariant, it follows that ∇' is also G–equivariant. For any point $z \in Z$, the restriction of ∇ to the fiber $(E_G)_z$ coincides with the Maurer–Cartan form on $(E_G)_z$. Therefore, for any point $x \in X$, the restriction of ∇' to the reduced fiber of E'_G over the point x coincides with the Maurer–Cartan form on the reduced fiber.

For the converse direction, if θ' is a $\mathfrak{g}$–valued holomorphic one–form on $E'_G := E_G/C$ satisfying the two conditions in the statement of the proposition, then the $\mathfrak{g}$–valued holomorphic one–form $q^*\theta'$ on E_G is clearly G–equivariant and it coincides with the Maurer–Cartan form on any fiber of E_G. Furthermore, since $q^*\theta'$ is pulled back from E_G/C, it vanishes on any orbit for the action of C on E_G. Therefore, $q^*\theta'$ is a holomorphic orbifold connection on E_G. This completes the proof of the Proposition. □

DEFINITION 3.6 Any holomorphic $\mathfrak{g}$–valued one–form on the total space of a ramified G–bundle E'_G satisfying the two conditions in Proposition 3.5 will be called a *holomorphic connection* on E'_G.

See [7] for properties of connections on ramified bundles.

4. Connections on a ramified bundle

We will first show that the second Chern class of the adjoint vector bundle of an orbifold G–bundle vanishes.

Proposition 4.1

Let E_G be an orbifold G–bundle over Z. Then $c_2(\mathrm{ad}(E_G)) \in H^4(Z, \mathbb{Q})$ vanishes.

Proof. Consider the orbifold vector bundle $\mathrm{ad}(E_G)$ over Z (the action of the elliptic curve C on E_G induces an action of C on $\mathrm{ad}(E_G)$). The orbifold vector bundle $\mathrm{ad}(E_G)$ defines a parabolic vector bundle over X with S as the parabolic divisor [6, Theorem 4.4]. Let W denote the holomorphic vector bundle over X underlying the parabolic vector bundle corresponding to $\mathrm{ad}(E_G)$.

Let $K(Z)$ denote the Grothendieck's K–group of coherent sheaves on Z. In $K(Z)$ we have

$$f^*(W \otimes_{\mathcal{O}_X} \mathcal{O}_X(S)) = \mathrm{ad}(E_G) + \sum_{i=1}^{h} V_i \in K(Z)\,, \tag{10}$$

where f is the projection in (1) and V_i is a vector bundle defined over the reduced curve $f^{-1}(p_i)_{\rm red} \subset Z$ (see [6, page 301, (19), (20)]). Here we consider a vector bundle E_0 defined over a curve $D_0 \subset Z$ as an element of $K(Z)$ by taking $\iota_* E_0$, where $\iota : D_0 \hookrightarrow Z$ is the inclusion map. Let L_i denote the restriction of the line bundle $\mathcal{O}_Z(f^{-1}(p_i)_{\rm red})$ to the reduced divisor $f^{-1}(p_i)_{\rm red}$. So by the Poincaré adjunction formula L_i is identified with the normal bundle of the smooth divisor $f^{-1}(p_i)_{\rm red} \subset Z$. Each vector bundle V_i over $f^{-1}(p_i)_{\rm red}$ in (10) can be taken to be of the form

$$V_i = \bigoplus_{j=1}^{n_i} L_i^{\otimes m_j}\,, \tag{11}$$

where n_i and m_j are arbitrary nonnegative integers (see [6, page 301, (19), (20)]).

If we fix a G–invariant nondegenerate bilinear form B on the Lie algebra $\mathfrak{g}$, then B gives a nondegenerate bilinear form on the fibers of the adjoint vector bundle $\mathrm{ad}(E_G)$. Therefore, $\mathrm{ad}(E_G)$ is isomorphic to $\mathrm{ad}(E_G)^*$, and in particular we have $c_1(\mathrm{ad}(E_G)) = 0$. Consequently,

$$c_2(\mathrm{ad}(E_G)) = -\mathrm{ch}_2(\mathrm{ad}(E_G))\,, \tag{12}$$

where ch_2 is the second Chern character.

Since the Chern character is additive, from (10) we have

$$\mathrm{ch}_2(f^*(W \otimes_{\mathcal{O}_X} \mathcal{O}_X(S))) = \mathrm{ch}_2(\mathrm{ad}(E_G)) + \sum_{i=1}^{h} \mathrm{ch}_2(V_i)\,. \tag{13}$$

Now, $\mathrm{ch}_2(f^*(W \otimes_{\mathcal{O}_X} \mathcal{O}_X(S))) = f^*\mathrm{ch}_2(W \otimes_{\mathcal{O}_X} \mathcal{O}_X(S)) = 0$ as X is a curve.

It is easy to see that the self–intersection number of the divisor $f^{-1}(p_i)_{\rm red} \subset Z$ vanishes. Indeed, if $x \neq y$, then $f^{-1}(x) \cap f^{-1}(y) = \emptyset$, while the curve $f^{-1}(x)$ is homologous to $f^{-1}(y)$. Therefore, the self–intersection number of the fiber $f^{-1}(x) \subset Z$ vanishes for each point $x \in X$. The self–intersection number of $f^{-1}(p_i)$ is m_i^2–times the self–intersection number of $f^{-1}(p_i)_{\rm red}$. Therefore, the self–intersection number of the divisor $f^{-1}(p_i)_{\rm red} \subset Z$ vanishes.

If D is a divisor on Z, then we have

$$\mathrm{ch}(\mathcal{O}_Z(D)) = 1 + [D] + D^2\,,$$

where $[D] \in H^2(Z, \mathbb{Q})$ is the cycle class of D and D^2 is the self–intersection number of D. We note that

$$L_i = \mathcal{O}_Z(f^{-1}(p_i)_{\rm red}) - \mathcal{O}_Z \in K(Z)\,,$$

where $L_i \in K(Z)$ are as in (11). Therefore, from the above observation that the self–intersection number of the divisor $f^{-1}(p_i)_{\rm red} \subset Z$ vanishes it follows immediately that $\mathrm{ch}_2(L_i^{\otimes k}) = 0$ for all $k \geq 0$. Now from (11) it follows that $\mathrm{ch}_2(V_i) = 0$ for all $i \in [1\,, h]$.

Consequently, from (13) we have

$$\mathrm{ch}_2(\mathrm{ad}(E_G)) = 0\,.$$

Therefore, from (12) we conclude that $c_2(\mathrm{ad}(E_G)) = 0$. This completes the proof of the Proposition. □

From now onwards, we will assume the group G to be simple.

Let E'_G be a polystable ramified G–bundle over X satisfying Condition B in Section 2 (see [4, Definition 3.13] for the definition of a polystable ramified G–bundle). Therefore, the parabolic G–bundle corresponding to E'_G is polystable (follows from the combination of [4, Theorem 3.14] and [3, Theorem 4.3]). Since E'_G satisfies the Condition B, the parabolic G–bundle corresponding to E'_G satisfies the Condition A, and hence there is a corresponding orbifold G–bundle over Z (see Theorem 2.1). Let E_G be the orbifold G–bundle over Z corresponding to E'_G.

Since the parabolic G–bundle corresponding to the ramified G–bundle E'_G is polystable, we conclude that the orbifold G–bundle E_G is orbifold polystable [6, Proposition 4.1], and hence the underlying G–bundle E_G is polystable in the usual sense [6, Proposition 4.3]. Now [1, Theorem 0.1] says that E_G admits a unique Einstein–Hermitian connection.

We will briefly recall the definition of a Einstein–Hermitian connection on E_G (see [1] for the details). Fix any maximal compact subgroup $K(G) \subset G$. Let $\mathfrak{z}(\mathfrak{g}) \subset \mathfrak{g}$ be the center of the Lie algebra. Note that

$$\mathfrak{z}(\mathfrak{g}) \subset H^0(Z, \mathrm{ad}(E_G)) .$$

Fix a Kähler form on Z. A connection ∇ on E_G is called a Einstein–Hermitian connection if there is a smooth reduction of structure group $E_{K(G)} \subset E_G$ of E_G to the subgroup $K(G) \subset G$ and a smooth connection $\nabla^{K(G)}$ on $E_{K(G)}$ such that following two conditions hold:

(1) the connection on E_G induced by the connection $\nabla^{K(G)}$ on $E_{K(G)}$ coincides with ∇, and

(2) there is an element $c \in \mathfrak{z}(\mathfrak{g})$ such that the curvature $K(\nabla^{K(G)})$ of the connection $\nabla^{K(G)}$, which is a $\mathrm{ad}(E_{K(G)})$–valued two–form over Z, satisfies the identity

$$\Lambda K(\nabla^{K(G)}) = c \in H^0(Z, \mathrm{ad}(E_G)) ,$$

where Λ is the adjoint of multiplication of differential forms by the Kähler form on Z.

Since G is simple and $c_2(\mathrm{ad}(E_G)) = 0$ (Proposition 4.1), the Einstein–Hermitian connection on E_G is flat (if the first and the second rational Chern classes of a vector bundle vanishes, then a Einstein–Hermitian connection on the vector bundle is flat [10, page 116, Corollary 4.13]).

Lemma 4.2

The Einstein–Hermitian connection on E_G is a flat holomorphic orbifold connection on the orbifold G–bundle E_G (see Definition 3.2).

Proof. Let ∇ denote the flat Einstein–Hermitian connection on E_G. From the uniqueness of the Einstein–Hermitian connection it follows that ∇ is preserved by the action of C on E_G. Indeed, for any point $c \in C$, the action of c on E_G gives an automorphism of E_G over the automorphism of Z given by the action of c. This automorphism of E_G will be denoted by $\tau(c)$. Consider the pullback of the connection form ∇ on E_G by the automorphism $\tau(c)$. This pulled back differential form $\tau(c)^*\nabla$ is an Einstein–Hermitian connection on $\phi(c)^*E_G = E_G$, where ϕ is the homomorphism in (2). From the uniqueness of the Einstein–Hermitian connection we conclude that $\phi(c)^*\nabla$ coincides with ∇. In other words, the action of C on E_G preserves ∇. So ∇ satisfies the first condition in Definition 3.2.

Take any point $x \in X \setminus S$. Let $E_G|_{f^{-1}(x)}$ be the restriction of the G–bundle E_G to the fiber $f^{-1}(x) \subset Z$, where f is defined in (1). Since the action of C on the fiber $f^{-1}(x)$ is free and transitive, the action of C on $E_G|_{f^{-1}(x)}$ gives a trivialization of the G–bundle $E_G|_{f^{-1}(x)}$. If we fix a point $z \in f^{-1}(x)$, then any fiber $(E_G)_y$, $y \in f^{-1}(x)$, is naturally identified with $(E_G)_z$ using the action of C on $E_G|_{f^{-1}(x)}$.

Since ∇ is a flat Einstein–Hermitian connection, the restriction of ∇ to $E_G|_{f^{-1}(x)}$ gives a flat Einstein–Hermitian connection on the G–bundle $E_G|_{f^{-1}(x)}$. Since $E_G|_{f^{-1}(x)}$ is a trivial G–bundle, the trivial connection on it gives a flat Einstein–Hermitian connection. Now from the uniqueness of the Einstein–Hermitian connection on $E_G|_{f^{-1}(x)}$ it follows that the restriction of ∇ to $E_G|_{f^{-1}(x)}$ coincides with the connection on $E_G|_{f^{-1}(x)}$ given by its trivialization. Thus ∇ satisfies the second condition in Definition 3.2. This completes the proof of the Lemma. □

Let

$$E'_G = E_G/C \tag{14}$$

be the ramified G–bundle corresponding to the polystable orbifold G–bundle E_G (see Theorem 2.1). Since the Einstein–Hermitian connection ∇ on E_G is a flat orbifold connection (Lemma 4.2), it gives a holomorphic connection on E'_G (see Definition 3.6 and Proposition 3.5). Let ∇' be the holomorphic connection on E'_G given by the flat Einstein–Hermitian connection ∇ on E_G.

Fix a proper parabolic subgroup

$$P \subset G$$

of the simple linear algebraic group G defined over $\mathbb{C}$. So the quotient G/P is a complete variety. For any character λ we have the associated line bundle

$$L(\lambda) := (G \times \mathbb{C}_\lambda)/P \tag{15}$$

over G/P; the action of P on $G \times \mathbb{C}_\lambda$ is defined as follows: any $p \in P$ sends any point $(g\,,c) \in G \times \mathbb{C}$ to $(gp\,,\lambda(p)^{-1}c)$.

Fix an antidominant character λ_0 of P such that the line bundle $L(\lambda_0)$ over G/P defined in (15) is ample. Let m be the least common multiple of the integers $\{m_1, \cdots, m_h\}$. Set

$$\lambda := \lambda_0^{\otimes m}\,. \tag{16}$$

Consider the quotient space E'_G/P, where E'_G is the ramified G–bundle in (14), which is a complete variety. Let

$$E'_G(\lambda) = (E'_G \times \mathbb{C}_\lambda)/P \tag{17}$$

be the quotient defined as follows: the action of any $p \in P$ sends any point $(z, c) \in E'_G \times \mathbb{C}$ to $(zp, \lambda(p)^{-1}c)$, where λ is the character of P defined in (16). Using the composition of maps

$$E'_G \times \mathbb{C}_\lambda \longrightarrow E'_G \longrightarrow E'_G/P$$

we obtain a projection of $E'_G(\lambda)$ (defined in (17)) to E'_G/P. We will show that $E'_G(\lambda)$ is an algebraic line bundle over the variety E'_G/P.

To show that first consider the quotient space E_G/P. Let

$$\xi(\lambda_0) := (E_G \times \mathbb{C}_{\lambda_0})/P$$

be the algebraic line bundle over E_G/P corresponding to the above character λ_0 of P (see (16)); the action of any $p \in P$ sends any point $(z, c) \in E_G \times \mathbb{C}$ to $(zp, \lambda_0(p)^{-1}c)$. Similarly, let

$$\xi(\lambda) := (E_G \times \mathbb{C}_\lambda)/P \tag{18}$$

be the algebraic line bundle over E_G/P corresponding to the character λ of P defined in (16). From (16) we conclude that

$$\xi(\lambda) = \xi(\lambda_0)^{\otimes m}. \tag{19}$$

The action of the elliptic curve C on E_G induces an action of C on E_G/P. Also note that the action of C on E_G induces an action of C on the line bundle $\xi(\lambda_0)$ (take the diagonal action of C on $E_G \times \mathbb{C}$ with C acting trivially on $\mathbb{C}$; this action descends to an action of C on the quotient space $\xi(\lambda_0) := (E_G \times \mathbb{C})/P$). Similarly, there is an induced action of C on the line bundle $\xi(\lambda)$. The projection from the total space of $\xi(\lambda_0)$ (respectively, $\xi(\lambda)$) to E_G/P commutes with the actions of C on E_G/P and $\xi(\lambda_0)$ (respectively, $\xi(\lambda)$). The order of the isotropy subgroup of any point $z \in Z$ for the action of C on Z is a submultiple of the integer m. Therefore, from (19) it follows immediately that the isotropy subgroup of any point $y \in (E_G/P)_z$ (for the action of C on E_G/P) acts trivially on the fiber $\xi(\lambda)_y$. This immediately implies that the quotient $\xi(\lambda)/C$ is an algebraic line bundle over $(E_G/P)/C$.

Clearly, $(E_G/P)/C = E'_G/P$, and the quotient space $\xi(\lambda)/C$ is identified with $E'_G(\lambda)$ defined in (17). Therefore, $E'_G(\lambda)$ is an algebraic line bundle over E'_G/P.

Theorem 4.3

The line bundle $E'_G(\lambda)$ over E'_G/P (defined in (17)*) is numerically effective, but it is not an ample line bundle.*

Proof. Fix a maximal compact subgroup $K(G) \subset G$ of the simple algebraic group G. As before, let ∇ denote the flat Einstein–Hermitian connection on the polystable orbifold G–bundle E_G over Z. Therefore, we have a C^∞ reduction of structure group of E_G to the maximal compact subgroup $K(G) \subset G$. Let

$$E_{K(G)} \subset E_G \tag{20}$$

be a C^∞ reduction of structure group of E_G to the subgroup $K(G)$ that supports a connection inducing the Einstein–Hermitian connection on E_G.

Set $K(P) := K(G) \cap P$. Note that the inclusion homomorphism $K(G) \hookrightarrow G$ induces an isomorphism of the quotient spaces

$$K(G)/K(P) = G/P\,.$$

So we have $E_G/P = E_{K(G)}/K(P)$, where $E_{K(G)}$ is constructed in (20). Consider the character λ of P and the line bundle $\xi(\lambda)$ over E_G/P associated by it (see (16) and (18) for their definitions). The line bundle $\xi(\lambda)$ is identified with the line bundle $(E_{K(G)} \times \mathbb{C}_\lambda)/K(P)$ over $E_{K(G)}/K(P)$; the quotient by $K(P)$ is defined as before (the action of any $g \in K(P)$ sends any $(z\,,c) \in E_{K(G)} \times \mathbb{C}$ to $(zg\,, \lambda(g)^{-1}c)$).

Note that $K(P)$ is a compact Lie group. In fact, $K(P)$ is a maximal compact subgroup of a Levi factor of P. Since $K(P)$ is compact, the character

$$\lambda\,:\, K(P) \longrightarrow \mathbb{C}^*$$

factors through the maximal compact subgroup $S^1 \subset \mathbb{C}^*$. Consequently, the associated line bundle $(E_{K(G)} \times \mathbb{C}_\lambda)/K(P)$ over $E_{K(G)}/K(P)$ has a natural Hermitian structure induced by the standard Hermitian structure on $\mathbb{C}$.

Using the earlier mentioned identification of the line bundle $\xi(\lambda)$ with the line bundle $(E_{K(G)} \times \mathbb{C}_\lambda)/K(P)$, the above Hermitian structure on $(E_{K(G)} \times \mathbb{C}_\lambda)/K(P)$ gives a Hermitian structure on $\xi(\lambda)$. Therefore, $\xi(\lambda)$ is a holomorphic Hermitian line bundle over E_G/P.

Let $\nabla^{\xi(\lambda)}$ denote the Chern connection on the holomorphic Hermitian line bundle $\xi(\lambda)$. Therefore,

$$\omega := \frac{\sqrt{-1}}{2\pi}(\nabla^{\xi(\lambda)})^2 \in \Omega^{1,1}(E_G/P) \tag{21}$$

is the Chern form that represents the first Chern class of $\xi(\lambda)$, where $(\nabla^{\xi(\lambda)})^2$ is the curvature of the connection $\nabla^{\xi(\lambda)}$. We will show that the form ω is nonnegative and its kernel at each point is of dimension two. For a $(1\,,1)$–form β defined on a complex manifold Y, the kernel of β at a point $y \in Y$ is the kernel of the homomorphism $T^{1,0}_y Y \longrightarrow (T^{0,1}_y Y)^*$ that sends any holomorphic tangent vector $v \in T^{1,0}_y Y$ to the contraction of $\beta(y)$ by v.

Take any point $z \in Z$. Fix a connected contractible analytic open subset $U \subset Z$ containing the point z. Let

$$q_z\,:\, U \longrightarrow z \tag{22}$$

be the projection map, and let $E_G^z := (E_G)_z$ be the G–bundle over the point z obtained by restricting E_G to z. Using the flat connection ∇ on E_G, the restriction $E_G|_U$ (to U) is canonically identified with $q_z^* E_G^z$ and the connection $\nabla|_U$ on $E_G|_U$ coincides with the natural connection on $q_z^* E_G^z$ (take parallel translations, for the flat connection ∇, along paths in U based at z to identify any other fiber with the fiber $(E_G)_z$). Let

$$f_z^0 \,:\, E_G|_U \longrightarrow E_G^z \tag{23}$$

be the G–equivariant projection obtained this way.

Since $G/P = K(G)/K(P)$, and the action of $K(P)$ on $\mathbb{C}$ defined using λ preserves the standard Hermitian structure on $\mathbb{C}$, we conclude that the holomorphic line bundle $L(\lambda)$ over G/P (defined in (15)) is equipped with a natural Hermitian structure. The line bundle $L(\lambda)$ is ample by our assumption on λ. Since the curvature form of the Chern connection on $L(\lambda)$ is invariant under the action of K on G/P, and $L(\lambda)$ is ample, it follows that $\frac{\sqrt{-1}}{2\pi}$–times the curvature of the Chern connection on $L(\lambda)$ is a positive form on G/P.

After fixing a point on the G–bundle $E_G^z := (E_G)_z$ over z, the trivial G–bundle $G \times \{z\}$ over the point z gets identified with the G–bundle E_G^z. This way the holomorphic Hermitian line bundle $L(\lambda)$ over G/P gets identified with the holomorphic Hermitian line bundle $\xi(\lambda)|_{(E_G/P)_z}$. Therefore, we conclude that $\frac{\sqrt{-1}}{2\pi}$–times the curvature of the Chern connection $\nabla^{\xi(\lambda)}|_{(E_G/P)_z}$ on $\xi(\lambda)|_{(E_G/P)_z}$ is a positive form on $(E_G/P)_z = (E_G)_z/P$.

Let $U' := E_G|_U \subset E_G$ be the open subset, where U is as in (22). Let $(E_G)_z)$. Let

$$f_z \,:\, E_G|_U/P = U'/P \longrightarrow E_G^z/P \tag{24}$$

be the projection obtained from the map f_z^0 in (23). The holomorphic Hermitian line bundle $\xi(\lambda)|_{U'/P}$ over $U'/P = (E_G|_U)/P$ is identified with the holomorphic Hermitian line bundle $f_z^* \xi(\lambda)|_{(E_G/P)_z}$, where f_z is the projection in (24). We already noted that $\frac{\sqrt{-1}}{2\pi}$–times the curvature of the Chern connection on $\xi(\lambda)|_{(E_G/P)_z}$ is a positive form on $(E_G/P)_z$. Since the pullback of a positive form is a nonnegative form, we conclude that $\frac{\sqrt{-1}}{2\pi}$–times the curvature of the Chern connection on $\xi(\lambda)|_{U'/P}$ is a nonnegative form. Since $\frac{\sqrt{-1}}{2\pi}$–times the curvature of the Chern connection on $\xi(\lambda)$ is the form ω (see (21)), we conclude that the restriction to $E_G|_U/P$ of the differential form ω (defined in (21)) is a nonnegative form.

Furthermore, for any point $y \in E_G|_U/P$, the kernel of $\omega(y)$ is precisely the two dimension subspace of the holomorphic tangent space $T_y E_G/P$ given by the horizontal subspace for the induced flat connection on the associated fiber bundle E_G/P; the connection ∇ on the G–bundle E_G over Z induces a connection on any fiber bundle associated to E_G, in particular on E_G/P, the fiber bundle associated to E_G for the left translation action of G on G/P. Indeed, the horizontal subspace at $y \in E_G|_U/P$ is precisely the kernel of the differential $\mathrm{d}f_z(y) \,:\, T_y E_G/P \longrightarrow T_{f_z(y)} E_G^z$, where f_z is defined in (24). The kernel of the pullback of a positive form coincides with the kernel of the differential.

Therefore, we conclude that the form ω is nonnegative. Furthermore, the kernel of ω coincides with the horizontal subbundle of the holomorphic tangent bundle TE_G/P.

Let

$$q_C \,:\, E_G/P \longrightarrow E'_G/P \,=\, (E_G/P)/C \tag{25}$$

be the quotient map. We recall that the line bundle $E'_G(\lambda)$ over E'_G/P is, by its definition, the line bundle $\xi(\lambda)/C$. Therefore, the pulled back line bundle $q_C^*E'_G(\lambda)$ over E_G/P is canonically identified with $\xi(\lambda)$. Since ω (defined in (21)) is a nonnegative form, the line bundle $\xi(\lambda)$ over E_G/P is numerically effective. Since $\xi(\lambda) \,=\, q_C^*E'_G(\lambda)$, and the map q_C in (25) is surjective, we conclude that the line bundle $E'_G(\lambda) \,=\, \xi(\lambda)/C$ over E'_G/P is numerically effective [8, Proposition 2.3].

Set $d_0 \,=\, \dim_{\mathbb{C}} E'_G/P \,=\, \dim_{\mathbb{C}} E_G/P - 1$. Since $\xi(\lambda) \,=\, q_C^*E'_G(\lambda)$, where q_C is the projection in (25), we conclude that

$$c_1(\xi(\lambda))^{d_0} \,=\, q_C^*c_1(E'_G(\lambda))^{d_0} \,\in\, H^{2d_0}(E_G/P,\, \mathbb{Q})\,. \tag{26}$$

Since the closed form ω in (21) represents the first Chern class $c_1(\xi(\lambda))$, the cohomology class $c_1(\xi(\lambda))^{d_0}$ is represented by the differential form $\bigwedge^{d_0}\omega$.

We saw earlier that the kernel of ω coincides with the subbundle of rank two of the holomorphic tangent bundle TE_G/P given by the horizontal subbundle (for the connection on the associated fiber bundle E_G/P induced by ∇). Since $d_0 \,=\, \dim_{\mathbb{C}} E_G/P - 1$, this immediately implies that the differential form $\bigwedge^{d_0}\omega$ on E_G/P vanishes identically. Therefore, from (26) we have $q_C^*c_1(E'_G(\lambda))^{d_0} \,=\, 0$. Since the homomorphism

$$q_C^* \,:\, \mathbb{Q} \,=\, H^{2d_0}(E'_G/P,\, \mathbb{Q}) \,\longrightarrow\, H^{2d_0}(E_G/P,\, \mathbb{Q})$$

is injective, we now conclude that

$$0 \,=\, c_1(E'_G(\lambda))^{d_0} \,\in\, H^{2d_0}(E'_G/P,\, \mathbb{Q}) \,=\, \mathbb{Q}\,.$$

This implies that the line bundle $E'_G(\lambda)$ is not ample (the first Chern class of an ample line bundle is positive and hence the top exterior power of the first Chern class is a positive number). This completes the proof of the Theorem. □

Let the polystable orbifold G–bundle E_G be such that the monodromy of the Einstein–Hermitian connection ∇ on E_G is dense in $K(G)$. We recall that fixing a point $z \in Z$ and also fixing a point $y \in (E_G)_z$ in the fiber, the monodromy of a flat Hermitian connection on E_G gives a homomorphism

$$\rho \,:\, \pi_1(Z\,, z) \,\longrightarrow\, K(G)$$

which is constructed using parallel translations of y along paths in Z starting at z. The condition that the image $\rho(\pi_1(Z\,, z))$ is dense in $K(G)$ does not depend on the choices of z and y. If the genus of the curve X is at least two, then the general polystable G–bundle satisfies the above condition that the monodromy of the Einstein–Hermitian connection on it is dense in $K(G)$.

Proposition 4.4

The line bundle $E'_G(\lambda)$ over E'_G/P has the property that for any proper closed subvariety $Y \subsetneq E'_G/P$, the restriction of $E'_G(\lambda)$ to Y is an ample line bundle.

Proof. In Theorem 4.3 we saw that the line bundle $E'_G(\lambda)$ over E'_G/P is numerically effective. Since $E'_G(\lambda)$ is numerically effective, for every closed subvariety $M \subset E'_G(\lambda)$ of (complex) dimension d we have

$$0 \leq c_1(E'_G(\lambda))^d \cap [M] \in \mathbb{Z}\,.$$

Using the Nakai–Moishezon criterion for ampleness (see [9, page 434, Theorem 5.1]), to prove that the restriction of $E'_G(\lambda)$ to any subvariety Y (as in the statement of the proposition) is ample it suffices to show that for any proper closed subvariety $M \subsetneq E'_G/P$ the inequality

$$c_1(E'_G(\lambda))^d \cap [M] > 0 \tag{27}$$

holds, where d is the (complex) dimension of M.

Since the Einstein–Hermitian connection ∇ on E_G is an orbifold connection (see Lemma 4.2), the Chern connection $\nabla^{\xi(\lambda)}$ (see (21)) on the line bundle $\xi(\lambda)$ over E_G/P descends to a Chern connection on the line bundle $E'_G(\lambda)$ over E'_G/P. The Hermitian structure on $\xi(\lambda)$ descends to a Hermitian structure on the line bundle $E'_G(\lambda)$ over E'_G/P. The descended Chern connection on $E'_G(\lambda)$ is the Chern connection for this descended Hermitian structure on $E'_G(\lambda)$. Let ∇'' denote the Chern connection on the line bundle $E'_G(\lambda)$ over E'_G/P obtained from the connection $\nabla^{\xi(\lambda)}$ on $\xi(\lambda)$.

Since the form ω in (21) is nonnegative (see the proof of Theorem 4.3), the form $\frac{\sqrt{-1}}{2\pi}(\nabla'')^2$ on E'_G/P is nonnegative, where $(\nabla'')^2$ is the curvature of the connection ∇''. Indeed, this follows immediately from the fact that

$$q_C^* \frac{\sqrt{-1}}{2\pi}(\nabla'')^2 = \omega\,,$$

where q_C is the projection in (25). Therefore, for any proper closed subvariety $M \subsetneq E'_G/P$ of (complex) dimension d we have

$$\int_M \left(\frac{\sqrt{-1}}{2\pi}(\nabla'')^2\right)^d \geq 0\,,$$

and furthermore, $\int_M (\frac{\sqrt{-1}}{2\pi}(\nabla'')^2)^d = 0$ if and only if the pullback of the differential form $(\frac{\sqrt{-1}}{2\pi}(\nabla'')^2)^d$ to M vanishes identically.

Assume that (27) fails. Let $M \subsetneq E'_G/P$ be a closed subvariety of complex dimension d such that $c_1(E'_G(\lambda))^d \cap [M] = 0$. Since

$$c_1(E'_G(\lambda))^d \cap [M] = \int_M \left(\frac{\sqrt{-1}}{2\pi}(\nabla'')^2\right)^d,$$

from the above observation we conclude that the pullback of the form $(\frac{\sqrt{-1}}{2\pi}(\nabla'')^2)^d$ to the subvariety M vanishes identically.

Let $y \in E_G/P$ be any point such that $q_C(y) \in M$, where q_C is the projection in (25). We have the differential of q_C

$$\mathrm{d}q_C(y) : T_y E_G/P \longrightarrow T_{q_C(y)} E'_G/P \tag{28}$$

between the holomorphic tangent spaces. The above observation that the pullback of the form $(\frac{\sqrt{-1}}{2\pi}(\nabla'')^2)^d$ to M vanishes identically implies that the subspace

$$T_{q_C(y)} M \subset T_{q_C(y)} E'_G/P$$

contains the image, by the homomorphism $\mathrm{d}q_C(y)$ (defined in (28)), of the horizontal subspace in $T_y E_G/P$ for the connection induced by ∇ on the associated fiber bundle E_G/P. The horizontal subspace of $T_y E_G/P$ is of dimension two. Note that the kernel of $\mathrm{d}q_C(y)$ is the direction along the orbit, passing through y, of the action of C on E_G/P. Therefore, the image, by the homomorphism $\mathrm{d}q_C(y)$, of the horizontal subspace of $T_y E_G/P$ is of dimension one.

Since $T_{q_C(y)} M$ contains the image of the horizontal subspace of $T_y E_G/P$, we conclude that the inverse image

$$\widehat{M} := q_C^{-1}(M) \subset E_G/P\,,$$

which is a subvariety of E_G/P, has the property that for any point $y \in q_C^{-1}(M) = \widehat{M}$, the tangent subspace

$$T_y \widehat{M} \subset T_y E_G/P$$

contains the two dimensional horizontal subspace of $T_y E_G/P$. This immediately implies that the subvariety $\widehat{M} \subset E_G/P$ is left invariant by the connection, induced by ∇, on the associated fiber bundle E_G/P.

Since the action of the maximal compact subgroup $K(G)$ on $G/P = K(G)/K(P)$ is transitive, if $\Gamma_0 \subset K(G)$ is a dense subgroup, then the action of Γ_0 on G/P does not preserve any proper closed subvariety of G/P. Since $\widehat{M} := q_C^{-1}(M) \subset E_G/P$ is left invariant by the connection, induced by ∇, on the associated fiber bundle E_G/P, we conclude that the subvariety $\widehat{M} \cap (E_G/P)_z \subset (E_G/P)_z$ is left invariant by the monodromy of the connection on E_G/P. Now the above remark — that a dense subgroup of $K(G)$ does not leave a proper closed subvariety of G/P invariant — combined with the assumption on E_G that the monodromy of ∇ is dense in $K(G)$ together imply that $\widehat{M} \cap (E_G/P)_z = (E_G/P)_z$. Therefore, we conclude that $q_C^{-1}(M) = E_G/P$. Hence $M = E'_G/P$. This contradicts the assumption that $M \neq E'_G/P$. Consequently, (27) is valid. This completes the proof of the Proposition. □

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